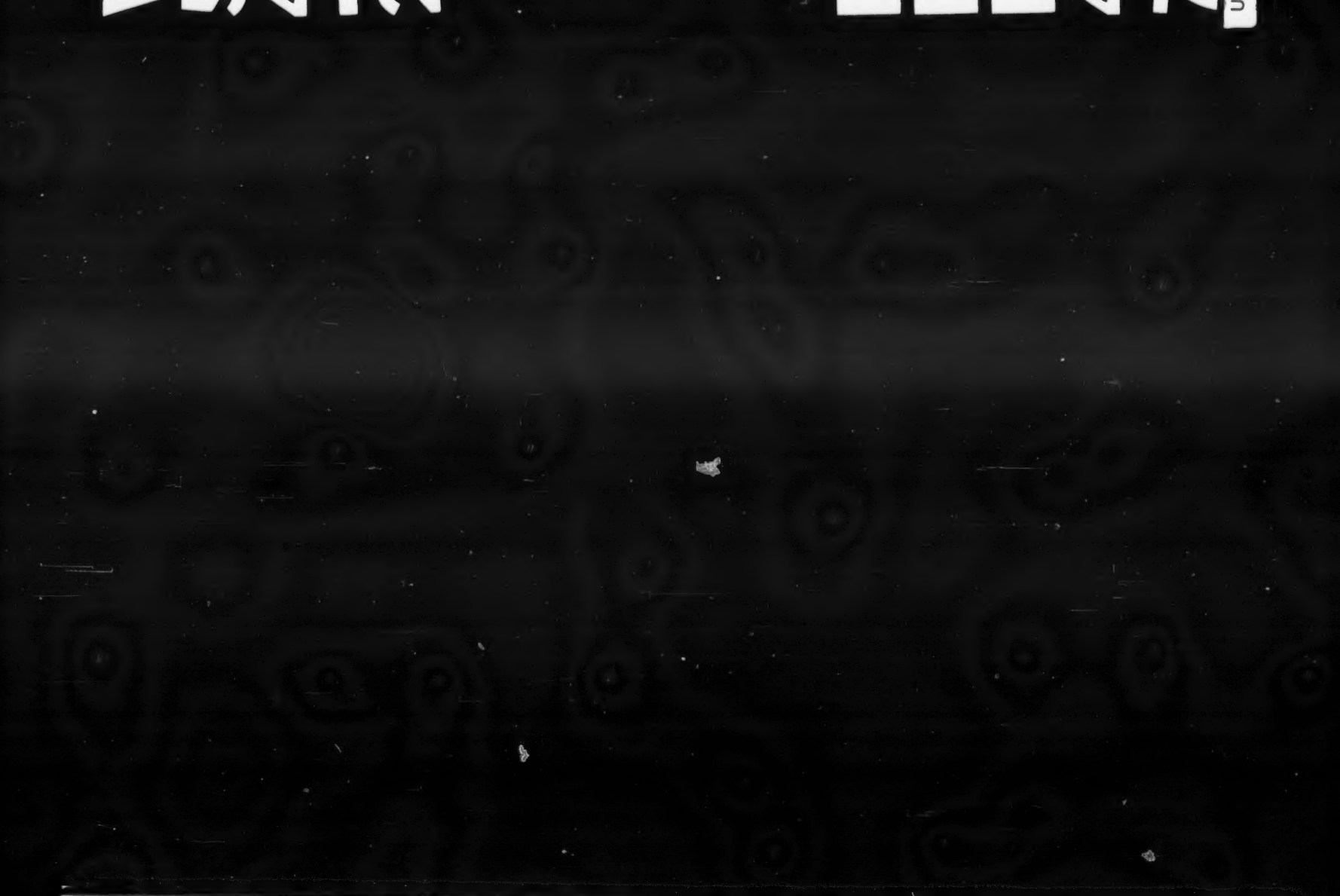

SOME IDENTITIES IN THE THEORY OF DETERMINANTS*

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"Cayley once said to me, in conversation, that if he had to give fifteen lectures on the whole of mathematics, he would devote *one* of them to determinants."—Felix Klein in *Elementary Mathematics from an Advanced Standpoint* [23, p. 143].

"Open the book where you will and you find yourself in the midst of a live question having vital connections at the present day with other live branches of mathematics. This is no small praise when one recalls the possibilities for barren formalism which the subject of determinants presents."—Maxime Bôcher in his review of Kowalewski's *Determinantentheorie* [10, p. 136].

"In fact, the importance of the concept of determinant has been, and currently is, vastly over-estimated. Systems of quations can be solved as easily and neatly without determinants as with, as is illustrated in Chapter I of this Monograph. In fact, perhaps ninety per cent of matric theory can be developed without mentioning a determinant. The concept is necessary in some places, however, and is very useful in many others, so one should not push this point too far."—C. C. MacDuffee in *Vectors and Matrices* [27, Introduction, p. v].

1. Introduction. Cayley considered determinants one of the central theories in all of mathematics; Bôcher emphasized chiefly their connections with other branches of mathematics; andMacDuffee points out that they are not so essential in some parts of algebra as they have long been considered. Mathematicians today, their judgment ripened by experience, probably would agree that the theory of determinants belongs to the class of tool theories and is not a subject of major interest for itself alone. Furthermore, the mathematics of today, almost completely dominated by algebra, topology, and similar abstract disciplines, presents but few situations in which determinants are needed as tools. The theory of determinants provides many powerful ones, however, and there is every reason to believe that it will yet play an important rôle in further developments in mathematics.

This exposition treats certain fundamental identities, knowledge of which is necessary for any extensive applications of determinants. Most of these identities concern determinants whose elements are themselves determinants. In particular, the paper treats the Laplace expansion of a determinant; the Binet-Cauchy multiplication theorem; Sylvester's theorem of 1839 and 1851; the Sylvester-Franke theorem; the Bazin-Reiss-Picquet theorem; the Cauchy, Jacobi, Franke, and Reiss theorems; and Sylvester's theorem on superdeterminants. A good modern proof of each of these except the first—which is too well known to require proof—will be given. The proofs emphasize matrix methods. The two key results are the Laplace expansion and the Sylvester-Franke theorem, and their proofs go back to first principles. No completely simple and direct proof of the Sylvester-Franke theorem is known, and discovery of one would be an event of real interest.

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Finally, the early history of these theorems is traced, and references are given to the original papers. Alternative proofs are indicated, and the interrelations of the theorems are pointed out.

2. Notation and definitions. The ordered array

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdots\cdots\cdots \\ a_{m1} & a_{m2} \cdots a_{mn} \end{pmatrix}, \tag{2.1}$$

with elements a_{ij} which are complex numbers, will be called a matrix; it will be indicated also by the briefer notation (a_{ij}), $i=1, \cdots, m$; $j=1, \cdots, n$. The transpose A' of A is the matrix whose rows are respectively the columns of A. The determinant whose elements are the elements in the square matrix A will be denoted by $|A|$ or $|a_{ij}|$.

Let M and N denote the sets of integers $1, 2, \cdots, m$ and $1, 2, \cdots, n$ respectively. Let $I^{(k)}$ denote a combination of k integers $i_1, i_2, \cdots, i_k$ selected from M, and let $I_i^{(k)}$, $i=1, 2, \cdots, C(m, k)$, be the complete set of combinations of k integers that can be formed from the m integers in M. Similarly, let $J_j^{(k)}$, $j=1, 2, \cdots, C(n, k)$, be the complete set of combinations of k integers $j_1, j_2, \cdots, j_k$ that can be formed from the n integers in N. The integers in both $I_i^{(k)}$ and $J_j^{(k)}$ are arranged in their natural order. It should be emphasized that the numbering of the combinations $I_i^{(k)}$ and $J_j^{(k)}$ is entirely arbitrary, but that a definite order is chosen now which will be used throughout the remainder of the paper. Finally, let $M-I_i^{(k)}$ (respectively $N-J_j^{(k)}$) denote the set of $m-k$ integers ($n-k$ integers), arranged in their natural order, in M (in N) but not in $I_i^{(k)}$ (not in $J_j^{(k)}$).

Next, let the submatrix of A formed by the k rows designated by the k integers in $I_i^{(k)}$ be denoted by $A(I_i^{(k)})$. Similarly, let the submatrix of A formed by the k columns designated by the k integers in $J_j^{(k)}$ be denoted by $A(J_j^{(k)})$. Finally, let $A(I_i^{(k)} \cdot J_j^{(k)})$ denote the submatrix of A which is contained in both $A(I_i^{(k)})$ and $A(J_j^{(k)})$. It follows that $A(I_i^{(k)} \cdot J_j^{(k)})$ is a square matrix with k rows and columns.

The kth compound of A, denoted by $A^{(k)}$, is defined for each integer k such that $1 \leqq k \leqq m$, $1 \leqq k \leqq n$, by

$$A^{(k)} = (|A(I_i^{(k)} \cdot J_j^{(k)})|), \quad i = 1, 2, \cdots, C(m, k); j = 1, 2, \cdots, C(n, k). \tag{2.2}$$

The element in the ith row and jth column of $A^{(k)}$ is thus the kth order determinant $|A(I_i^{(k)} \cdot J_j^{(k)})|$ of a submatrix of A.

When A is a square matrix with n rows and columns, $A(N-I_i^{(k)} \cdot N-J_j^{(k)})$ will denote the submatrix of A which is complementary to $A(I_i^{(k)} \cdot J_j^{(k)})$; it is obtained from A by striking out the rows designated by the integers in $I_i^{(k)}$ and the columns designated by the integers in $J_j^{(k)}$. For the sake of completeness and symmetry, $A(N-I_1^{(n)} \cdot N-J_1^{(n)})$ will be defined to be the matrix with the single

element 1. Thus when A is a square matrix with n rows and columns, the kth adjoint compound of A, denoted by $\text{adj}^{(k)} A$, is defined for each k such that $1 \leqq k \leqq n$ by

$$\text{adj}^{(k)} A = ((-1)^{s_i+s_j} \mid A(N - I_i^{(k)} \cdot N - J_j^{(k)}) \mid)', \qquad i = 1, 2, \cdots, C(n, k); j = 1, 2, \cdots, C(n, k). \tag{2.3}$$

Here s_i and s_j are the sums of the k integers in $I_i^{(k)}$ and $J_j^{(k)}$, respectively. Thus $\text{adj}^{(k)} A$ is a compound matrix whose elements are signed determinants with $(n-k)$ rows and columns. Note the prime which indicates that the matrix in the right member of (2.3) is to be transposed. It may be observed that $\text{adj}^{(1)} A$ is the matrix usually denoted* by adj A and called the adjoint of A.

When A and B are any two square matrices, $A[B(J_i^{(k)})/A(J_j^{(k)})]$, or $A[B_i^{(k)}/A_j^{(k)}]$ for short, will be used to denote the matrix obtained by replacing in order the k columns $A(J_j^{(k)})$ in A by the k columns $B(J_i^{(k)})$ in B.

Finally, notation will be required for bordered matrices. Let $R^{(h)}$ and $S^{(h)}$ denote the set of integers $1, 2, \cdots, h$ in N. Let $U_i^{(k)}$, $i=1, 2, \cdots, C(n-h, k)$ (respectively $V_j^{(k)}$, $j=1, 2, \cdots, C(n-h, k)$) denote the complete set of combinations of k integers each that can be selected from the sets $N-R^{(h)}(N-S^{(h)})$. Then $A(R^{(h)}+U_i^{(k)} \cdot S^{(h)}+V_j^{(k)})$, $i, j=1, 2, \cdots, C(n-h, k)$, will denote all the submatrices of A that can be formed by bordering $A(R^{(h)} \cdot S^{(h)})$ with k of the remaining rows and columns.

3. Laplace's expansion. Let $|A|$ be any nth order determinant and k any integer such that $1 \leqq k \leqq n$. Then the Laplace expansion of $|A|$ is

$$|A| = \sum (-1)^{s_i+s_j} |A(I_i^{(k)} \cdot J_j^{(k)})| \, |A(N - I_i^{(k)} \cdot N - J_j^{(k)})|, \tag{3.1}$$

where s_i and s_j are defined in (2.3). The summation extends either over the set $i=1, 2, \cdots, C(n, k)$ with j arbitrary, or over the set $j=1, 2, \cdots, C(n, k)$ with i arbitrary. This result is too well known to require proof here (see [1, pp. 78–81], [9, pp. 24–26], [24, pp. 34–36]).

The following formulas follow from Laplace's expansion:

$$A^{(k)} \text{adj}^{(k)} A = \text{adj}^{(k)} A A^{(k)} = |A| I, \tag{3.2}$$

$$|A^{(k)} \text{adj}^{(k)} A| = |\text{adj}^{(k)} A A^{(k)}| = |A|^{C(n,k)}. \tag{3.3}$$

Here A is any square matrix with n rows and columns and I is the identity matrix.

4. The Binet-Cauchy multiplication theorem. The multiplication theorem provides an evaluation of the determinant of the product of two matrices.

THEOREM. *Let A and B be two matrices with m rows and n columns, and let*

* The notations $A^{(k)}$ and $\text{adj}^{(k)} A$ are standard; see, for example, Aitken [1, pp. 90–91] and Albert [2, p. 57]. Aitken, along with many writers in the British Isles, uses *adjugate* where *adjoint* is commonly used in America.

AB' denote the row by column matrix product of A and B'. Then

$$|AB'| = 0 \qquad m > n, \tag{4.1}$$

$$= |A||B| \qquad m = n, \tag{4.2}$$

$$= \sum_{j=1}^{C(n,m)} |A(J_j^{(m)})||B(J_j^{(m)})| \qquad m < n. \tag{4.3}$$

Although there are at least two standard proofs of this theorem (see [1, pp. 80–81, 85–87], [24, pp. 60–70]), the most elegant one is based on the following obvious identity for partitioned matrices:

$$\begin{pmatrix} I & A \\ O & I \end{pmatrix}\begin{pmatrix} A & O \\ -I & B' \end{pmatrix} = \begin{pmatrix} O & AB' \\ -I & B' \end{pmatrix},$$

where I is the identity matrix and O is the zero matrix. Multiplication by the first factor on the left of this equation replaces each row of the other factor by itself plus certain multiples of other rows. Since these operations do not change the value of a determinant,

$$\begin{vmatrix} A & O \\ -I & B' \end{vmatrix} = \begin{vmatrix} O & AB' \\ -I & B' \end{vmatrix}. \tag{4.4}$$

Next, evaluate these two determinants by means of Laplace's expansion. The determinant on the right in all cases is $(-1)^{mn+n}|AB'|$. The one on the left is clearly $|A|\ |B'|$ if $m=n$, but in the other cases it must be examined more closely. Written out in full, this determinant is

$$\begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} & 0 & 0 \cdots 0 \\ a_{21} & a_{22} \cdots a_{2n} & 0 & 0 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} \cdots a_{mn} & 0 & 0 \cdots 0 \\ -1 & 0 \cdots 0 & b_{11} & b_{21} \cdots b_{m1} \\ 0 & -1 \cdots 0 & b_{12} & b_{22} \cdots b_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 \cdots -1 & b_{1n} & b_{2n} \cdots b_{mn} \end{vmatrix}.$$

If $m>n$, an expansion of this determinant by Laplace's method, using minors of m rows and columns selected from the first m rows, shows that its value is zero; (4.1) follows. Exactly the same expansion is used when $m \leqq n$. Consider a term of the expansion in which the first factor is one of the determinants $|A(J_j^{(m)})|$. The second factor can be expanded by Laplace's method in terms of minors of m rows and columns formed from the last m columns. There is a single term in this expansion, and its value is $|[B(J_j^{(m)})]'|$, or $|B(J_j^{(m)})|$, except perhaps for sign. A calculation of the signs shows that the complete product which

includes the factor $|A(J_j^{(m)})|$ is $(-1)^{mn+n}|A(J_j^{(m)})||B(J_j^{(m)})|$. All terms in the expansion are zero except those for which the first factor is one of the determinants $|A(J_j^{(m)})|$. Thus, when $m \leqq n$, the relation

$$\sum_{j=1}^{C(n,m)} (-1)^{mn+n}|A(J_j^{(m)})||B(J_j^{(m)})| = (-1)^{mn+n}|AB'|$$

is obtained by equating the evaluations of the two sides of (4.4). Equations (4.2) and (4.3) follow from this relation, and the proof is complete.

An important extension of the multiplication theorem concerns the kth compound of the product of two matrices.

THEOREM. *Let A be a matrix with m rows and n columns, and let B be a matrix with n rows and m columns. If k is any integer such that $1 \leqq k \leqq m$ and $1 \leqq k \leqq n$, then*

$$|(AB)^{(k)}| = |A^{(k)}B^{(k)}|, \tag{4.5}$$

$$|AB|^{C(m-1,k-1)} = |A^{(k)}B^{(k)}|. \tag{4.6}$$

The proof of (4.5) follows from the matrix identities

$$\begin{aligned}(AB)^{(k)} &= (|A(I_i^{(k)})B(J_j^{(k)})|) \\ &= \left(\sum_{t=1}^{C(n,k)} |A(I_i^{(k)}\cdot J_t^{(k)})||B(I_t^{(k)}\cdot J_j^{(k)})|\right) \\ &= (|A(I_i^{(k)}\cdot J_j^{(k)})|)(|B(I_i^{(k)}\cdot J_j^{(k)})|) \\ &= A^{(k)}B^{(k)}.\end{aligned} \tag{4.7}$$

The first equality here results from observing that the element in the ith row and jth column of $(AB)^{(k)}$ is the determinant of the matrix obtained by multiplying the ith combination $A(I_i^{(k)})$ of rows of A into the jth combination $B(J_j^{(k)})$ of columns of B. The second equality follows from an application of (4.3). The third equality results from the definition of matrix multiplication. The fourth equality results from the definition of the kth compound. The proof of (4.5) now follows by equating the determinants of the first and last arrays.

The proof of (4.6) follows from the Sylvester-Franke theorem that $|(AB)^{(k)}| = |AB|^{C(m-1,k-1)}$ (see (6.1) below). Although the proof will not be complete until the Sylvester-Franke theorem has been proved, the result was included here so that the statement of the multiplication theorem would be complete.

The proof of (4.7) is a proof of the following important theorem in matrix theory: the kth compound of the product of two matrices is equal to the product of the kth compounds of these matrices.

The multiplication theorem has many important applications. For example, the $(m-1)$ measure $(P_1P_2 \cdots P_m)$ of the $(m-1)$ cell determined by the points $P_i:(x_{i1}, x_{i2}, \cdots, x_{in})$, $i=1, 2, \cdots, m$, in n-space is known [8, pp. 293–296] to be $|AA'|^{\frac{1}{2}}/(m-1)!$, where A is the matrix whose ith row is

$$(x_{i1} - x_{m1}, x_{i2} - x_{m2}, \cdots, x_{in} - x_{mn}), \qquad i = 1, 2, \cdots, (m-1).$$

An application of (4.3) yields the formulas of analytic geometry for distance, area, and so on in the following form:

$$(P_1P_2) = \frac{1}{1!}\left[\begin{vmatrix} x_{11} & 1 \\ x_{21} & 1 \end{vmatrix}^2 + \begin{vmatrix} x_{12} & 1 \\ x_{22} & 1 \end{vmatrix}^2 + \cdots + \begin{vmatrix} x_{1n} & 1 \\ x_{2n} & 1 \end{vmatrix}^2\right]^{1/2},$$

$$(P_1P_2P_3) = \frac{1}{2!}\left[\begin{vmatrix} x_{11} & x_{12} & 1 \\ x_{21} & x_{22} & 1 \\ x_{31} & x_{32} & 1 \end{vmatrix}^2 + \begin{vmatrix} x_{11} & x_{13} & 1 \\ x_{21} & x_{23} & 1 \\ x_{31} & x_{33} & 1 \end{vmatrix}^2 + \cdots + \begin{vmatrix} x_{1n-1} & x_{1n} & 1 \\ x_{2n-1} & x_{2n} & 1 \\ x_{3n-1} & x_{3n} & 1 \end{vmatrix}^2\right]^{1/2},$$

and so on. The identity (4.3) in the case $m=2$, $n=3$ is the familiar Lagrange's identity of vector analysis. Again, if A is any matrix whose elements are real, the determinant $|AA'|$ is known **[14, pp. 29–30]** as Gram's determinant. It follows from (4.3) that $|AA'| \geqq 0$. When A is a matrix with two rows, this inequality is Schwarz's inequality. A necessary and sufficient condition that the m vectors which form the m rows of A be linearly independent is $|AA'| > 0$.

5. Sylvester's theorem of 1839 and 1851. This theorem gives an important series of representations for the product of two determinants.

THEOREM. *Let A and B be any two square matrices with n rows and columns, and let k be any integer such that $1 \leqq k \leqq n$. Then*

$$(5.1)\qquad \sum_{i=1}^{C(n,k)} |A[B(J_i^{(k)})/A(J_i^{(k)})]|\,|B[A(J_i^{(k)})/B(J_i^{(k)})]| = 0, \qquad i \neq j,$$

$$= |A|\,|B|, \qquad i = j.$$

The most elegant proof of this theorem **[16, vol. II, pp. 73–74]** is obtained by equating corresponding elements in two evaluations of the matrix product $\mathrm{adj}^{(k)} AB^{(k)}\ \mathrm{adj}^{(k)} BA^{(k)}$. First recall (3.2) and the fact that matrix multiplication is associative. Then

$$\begin{aligned} \mathrm{adj}^{(k)}AB^{(k)}\mathrm{adj}^{(k)}BA^{(k)} &= \mathrm{adj}^{(k)}A \cdot B^{(k)}\mathrm{adj}^{(k)}B \cdot A^{(k)} \\ &= \mathrm{adj}^{(k)}A \cdot |B|\, I \cdot A^{(k)} \\ &= \mathrm{adj}^{(k)}AA^{(k)} \cdot |B|\, I \\ &= |A|\,|B|\, I. \end{aligned}$$

Again,

$$\begin{aligned} &\mathrm{adj}^{(k)}AB^{(k)}\mathrm{adj}^{(k)}BA^{(k)} \\ &\qquad = \mathrm{adj}^{(k)}AB^{(k)} \cdot \mathrm{adj}^{(k)}BA^{(k)} \\ &\qquad = (|A[B(J_i^{(k)})/A(J_i^{(k)})]|)(|B[A(J_i^{(k)})/B(J_i^{(k)})]|) \\ &\qquad = \left(\sum_{i=1}^{C(n,k)} |A[B(J_i^{(k)})/A(J_i^{(k)})]|\,|B[A(J_i^{(k)})/B(J_i^{(k)})]|\right). \end{aligned}$$

In the matrices on the right, the element in the ith row and the jth column is shown in each case. Equations (5.1) follow by equating corresponding elements in the two evaluations of the matrix product. The proof is complete.

6. The Sylvester-Franke theorem. This theorem evaluates $|A^{(k)}|$ in terms of $|A|$.

THEOREM. *Let A be any square matrix with n rows and columns. Then*

$$|A^{(k)}| = |A|^{C(n-1,k-1)} \qquad k = 1, 2, \cdots, n. \tag{6.1}$$

A proof **[24**, pp. 94–98**]** by mathematical induction on n will be given. The theorem is obviously true for $n=2$. It will be shown that it is true for all determinants of order n if it is true for all determinants of order $n-1$.

Consider a_{nn} as a variable and all other elements in A as constants. Then $|A|$ and $|A^{(k)}|$ are functions of a_{nn}, and $|A| = Da_{nn}+E$. Here D is the minor of a_{nn} in $|A|$ and E is the determinant obtained by replacing a_{nn} in $|A|$ by 0. There is no loss in generality in assuming that the elements in $A^{(k)}$ are ordered in such a way that a_{nn} occurs only in the submatrix

$$(|A(I_i^{(k)} \cdot J_j^{(k)})|) \qquad i, j = 1, 2, \cdots, r, \tag{6.2}$$

where $r = C(n-1, k-1)$. Furthermore, $|A(I_i^{(k)} \cdot J_j^{(k)})| = d_{ij}a_{nn}+e_{ij}$, $i, j = 1, 2, \cdots, r$, where d_{ij} is a $(k-1)$ rowed minor of D and e_{ij} is a determinant similar to E. It follows from the induction hypothesis, namely, the hypothesis that the theorem is true for all determinants of order $(n-1)$, that

$$|d_{ij}| = D^{C(n-2,k-2)} \qquad i, j = 1, 2, \cdots, r.$$

Then the determinant of the matrix in (6.2) is the following polynomial in a_{nn} of degree r:

$$D^{C(n-2,k-2)}a_{nn}^r + \cdots. \tag{6.3}$$

Next, observe that

$$(|A(I_i^{(k)} \cdot J_j^{(k)})|) \qquad i, j = r+1, r+2, \cdots, C(n, k), \tag{6.4}$$

is the matrix in $A^{(k)}$ complementary to the one in (6.2), and that it is the kth compound of the matrix in D. Then by the induction hypothesis, the determinant of the matrix in (6.4) is $D^{C(n-2,k-1)}$.

Now use Laplace's expansion to expand $|A^{(k)}|$ by minors formed from the first r rows. It follows from results already established that the first term in the expansion is $D^{C(n-2,k-2)+C(n-2,k-1)}a_{nn}^r + \cdots$, a polynomial in a_{nn} of degree r. The other terms in the expansion are polynomials in a_{nn} of degree less than r. Since $C(n-2, k-2)+C(n-2, k-1) = C(n-1, k-1) = r$, it follows that

$$|A^{(k)}| = D^r a_{nn}^r + \cdots. \tag{6.5}$$

Furthermore, by (3.3) and (4.2)

$$|A^{(k)}|\,|\operatorname{adj}^{(k)}A| = |A|^{C(n,k)} = (Da_{nn}+E)^{C(n,k)}. \tag{6.6}$$

Thus every value of a_{nn} for which $|A^{(k)}|$ vanishes is a zero of $|A|^{C(n,k)}$. But the latter, equal to $(Da_{nn}+E)^{C(n,k)}$, vanishes only for $a_{nn}=-E/D$. Thus all zeros of the polynomial in (6.5) are $a_{nn}=-E/D$, and

$$|A^{(k)}| = D^r(a_{nn}+E/D)^r = (Da_{nn}+E)^r = |A|^r = |A|^{C(n-1,k-1)}.$$

Since the theorem is true for all determinants of the second order, the proof of the theorem follows by a complete induction.

It may be remarked that the proof given above can be varied slightly so that the Fundamental Theorem of Algebra is not required [24, pp. 97–98]. The proof has assumed $D\neq 0$, but the usual methods can be used to show that the result holds without this restriction.

COROLLARY. *Let A be any square matrix with n rows and columns. Then*

$$|\operatorname{adj}^{(k)}A| = |A|^{C(n-1,k)} \qquad k = 1, 2, \cdots, n. \tag{6.7}$$

The proof follows from (6.6) and (6.1). A more instructive proof, however, is obtained by showing that $|\operatorname{adj}^{(k)}A| = |A^{(n-k)}|$, $k=1, 2, \cdots, (n-1)$, and recalling the convention that $|\operatorname{adj}^{(n)}A| = 1$. Assume $1\leqq k\leqq n-1$. From the ith row of $|[\operatorname{adj}^{(k)}A]'|$ factor out $(-1)^{s_i}$ for $i=1, 2, \cdots, C(n, k)$. Similarly, from the jth column factor out $(-1)^{s_j}$ for $j=1, 2, \cdots, C(n, k)$. Since

$$\sum_{i=1}^{C(n,k)} s_i + \sum_{j=1}^{C(n,k)} s_j = 2\sum_{i=1}^{C(n,k)} s_i$$

and is thus an even number, the value of the determinant is unchanged. The new determinant is $|A^{(n-k)}|$. Then (6.7) follows from (6.1).

7. The Bazin-Reiss-Picquet theorem. Recall the meaning of $B[A(J_i^{(k)})/B(J_j^{(k)})]$, or $B[A_i^{(k)}/B_j^{(k)}]$ for short, from section 2.

THEOREM. *Let A and B be any two square matrices with n rows and columns, and let k be any integer such that $1\leqq k\leqq n$. Then*

$$|A|^{C(n-1,k-1)}\,|B|^{C(n-1,k)} = ||B[A_j^{(k)}/B_i^{(k)}]||, \tag{7.1}$$

where $i, j=1, 2, \cdots, C(n, k)$ and the element shown in the determinant on the right is the one in the ith row and jth column.

The proof once more is based on a matrix identity. An application of Laplace's expansion shows that

$$\operatorname{adj}^{(k}BA^{(k)} = (|B[A(J_j^{(k)})/B(J_i^{(k)})]|) \tag{7.2}$$

where $i, j=1, 2, \cdots, C(n, k)$. The proof is completed by equating the determinants of the matrices on the two sides of (7.2) and applying the multiplication theorem (4.2), the Sylvester-Franke theorem (6.1) and the corollary (6.7) to the left side.

It should be observed that the proof of (7.1) was based on the Sylvester-Franke theorem. There is a second proof **[37]** in the case $k=1$ which, although not so elegant, is instructive. Furthermore, it depends on nothing more than Cramer's Rule and the multiplication theorem. Assume $|B| \neq 0$ and express each column of A as a linear combination of the columns of B so that $A=BC$. Then $|A|=|B||C|$ by the multiplication theorem. Calculate the elements of C by Cramer's Rule and substitute in the last equation; the result is (7.1) in the case $k=1$. The restriction $|B| \neq 0$ can be removed in the usual way. This proof does not require the result $|\text{adj } B| = |B|^{n-1}$, which is the simplest case of the Sylvester-Franke theorem.

Many important relations are corollaries of the identity (7.1). For example, if $A=I$ and $k=1$, then (7.1) becomes

$$|B|^{n-1} = |\text{adj } B|. \tag{7.3}$$

By specializing A and B properly, Jacobi's theorem can be obtained **[48]** from (7.1) in the case $k=1$. If

$$A = \begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix}, \qquad B = \begin{pmatrix} A_1 & O \\ A_2 & I \end{pmatrix},$$

where $A_1=(a_{ij})$, $i, j=1, 2, \cdots, h$, then (7.1) with $k=1$ can be simplified to give the following identity, which is the simplest case of Sylvester's theorem on superdeterminants (see **[24**, pp. 76–80, 90–94, 100–101] and [1, pp. 45–50]):

$$|A||A_1|^{n-h-1} = ||A(R^{(h)} + U_i^{(1)} \cdot S^{(h)} + V_j^{(1)})||, \tag{7.4}$$

$i, j=1, 2, \cdots, n-h$. The element shown in the determinant on the right is the one in the ith row and jth column. The identity (7.4) is the basis of the standard method of evaluating determinants by reducing them to equal determinants of lower order. If A_1 has a single element, then Sylvester's theorem (7.4) evaluates the nth order determinant $|A|$ in terms of a determinant of order $(n-1)$, each element of which is a determinant of order 2. If A_1 has two rows and columns, the order of $|A|$ is reduced by 2, but each element of the new determinant is a determinant of order 3; and so on.

Many special cases of the identity (7.1) are listed by Cullis **[16**, vol. II, pp. 63–65].

8. Jacobi's, Franke's and Reiss' theorems. The theorems known by the names of Jacobi, Franke, and Reiss evaluate the minors in certain compound determinants. Reiss' theorem includes Franke's, which in turn includes Jacobi's; hence, only Reiss' theorem need be proved. The most elegant proof once more is based on certain matrix identities **[1, pp. 97–103]**.

THEOREM. *Let A and B be any square matrices with n rows and columns. Then*

$$(8.1)\quad [\mathrm{adj}^{(k)}B\cdot A^{(k)}]^{(h)} = |A|^{h-C(n-1,k)}\,|B|^{h-C(n-1,k-1)}\mathrm{adj}^{(h)}[\mathrm{adj}^{(k)}A\cdot B^{(k)}].$$

The proof begins with the obvious identity

$$(8.2)\qquad \mathrm{adj}^{(k)}A\cdot B^{(k)}\mathrm{adj}^{(k)}B\cdot A^{(k)} = |AB|\,I.$$

The hth compounds of the matrices on the two sides of this equation are equal. If (4.7) is applied to the left side written in the form $\mathrm{adj}^{(k)}\,AB^{(k)}\cdot\mathrm{adj}^{(k)}\,BA^{(k)}$, it follows that

$$(8.3)\qquad [\mathrm{adj}^{(k)}A\cdot B^{(k)}]^{(h)}[\mathrm{adj}^{(k)}B\cdot A^{(k)}]^{(h)} = |AB|^{h}I.$$

Next, it follows from (3.3) that

$$(8.4)\qquad [\mathrm{adj}^{(k)}A\cdot B^{(k)}]^{(h)}\mathrm{adj}^{(h)}[\mathrm{adj}^{(k)}A\cdot B^{(k)}] = |\,\mathrm{adj}^{(k)}A\cdot B^{(k)}|\,I.$$

If $[\mathrm{adj}^{(k)}\,A\cdot B^{(k)}]$ is a non-singular matrix, it follows from the Sylvester-Franke theorem (6.1) that its hth compound is also non-singular; in this case, $[\mathrm{adj}^{(k)}\,A\cdot B^{(k)}]^{(h)}$ has a reciprocal matrix. From (8.3) and (8.4) it follows that

$$|AB|^{-h}[\mathrm{adj}^{(k)}A\cdot B^{(k)}]^{(h)}[\mathrm{adj}^{(k)}B\cdot A^{(k)}]^{(h)}$$
$$= |\,\mathrm{adj}^{(k)}A\cdot B^{(k)}|^{-1}[\mathrm{adj}^{(k)}A\cdot B^{(k)}]^{(h)}\mathrm{adj}^{(h)}[\mathrm{adj}^{(k)}A\cdot B^{(k)}].$$

Multiply the two sides of this equation by $|AB|^{h}$ and then by the reciprocal of $[\mathrm{adj}^{(k)}\,A\cdot B^{(k)}]^{(h)}$. The resulting relation is

$$(8.5)\quad [\mathrm{adj}^{(k)}B\cdot A^{(k)}]^{(h)} = |AB|^{h}\,|\,\mathrm{adj}^{(k)}A\cdot B^{(k)}|^{-1}\mathrm{adj}^{(h)}[\mathrm{adj}^{(k)}A\cdot B^{(k)}].$$

Apply the multiplication theorem and the Sylvester-Franke theorem to the two determinants on the right side of (8.5); the identity (8.1) results. It is shown in the usual way that (8.1) holds even if $[\mathrm{adj}^{(k)}\,A\cdot B^{(k)}]$ is a singular matrix. The proof is complete.

If $k=1$ and $A=I$, the identity (8.1) is

$$(8.6)\qquad [\mathrm{adj}\,B]^{(h)} = |B|^{h-1}\mathrm{adj}^{(h)}B.$$

The determinant identities obtained by equating corresponding elements in the matrices on the two sides of this identity are known as Jacobi's theorem. If $h=n$, (8.6) becomes (7.3), a result sometimes known as Cauchy's theorem.

If $B=I$, the identity (8.1) becomes

$$(8.7)\qquad [A^{(k)}]^{(h)} = |A|^{h-C(n-1,k)}\mathrm{adj}^{(h)}[\mathrm{adj}^{(k)}A].$$

The determinant identities obtained by equating corresponding elements in the matrices on the two sides of this identity are known as Franke's theorem. If $h=C(n,k)$, relation (8.7) becomes the Sylvester-Franke theorem (6.1).

9. Sylvester's theorem on superdeterminants. A special case of Sylvester's theorem has been given in (7.4). The theorems proved in the last section can now be used to prove the general theorem.

THEOREM. *Let A be any matrix with n rows and columns, and let $A(R^{(h)} \cdot S^{(h)})$ be the submatrix with h rows and columns, $0 \leqq h \leqq n-1$, in the upper left hand corner. Let k be an integer such that $1 \leqq k \leqq n-h$. Then*

$$(9.1)\quad |A|^{C(n-h-1,k-1)} \, |A(R^{(h)} \cdot S^{(h)})|^{C(n-h-1,k)} = ||A(R^{(h)} + U_i^{(k)} \cdot S^{(h)} + V_j^{(k)})||,$$

where $i, j = 1, 2, \cdots, C(n-h, k)$ and the element shown in the determinant on the right is the one in the ith row and the jth column.

To prove the theorem [24, pp. 100–101], consider first the determinant

$$(9.2)\qquad D = |\alpha_{ij}| \qquad i, j = h+1, h+2, \cdots, n,$$

where α_{ij} is the cofactor of a_{ij} in $|A|$. Thus D is a minor of the transpose of adj A. Let E denote the determinant which forms the right member of (9.1). By Jacobi's theorem (8.6) any $(n-h-k)$ rowed minor of D is equal to one of the superdeterminants in E, properly signed with $+$ or $-$, and multiplied by $|A|^{n-h-k-1}$. Then

$$(9.3)\qquad |A|^{(n-h-k-1)C(n-h,k)} E = D^{C(n-h-1,n-h-k-1)} = D^{C(n-h-1,k)}$$

by the Sylvester-Franke theorem (6.1) since the left side is equal to the determinant of the $(n-h-k)$ rowed minors of D. In making this statement, it must be observed that the value of the compound determinant is the same whether it be formed with signed or unsigned minors (see the proof of (6.7)). By Jacobi's theorem (8.6)

$$(9.4)\qquad D = |A|^{(n-h-1)} \, |A(R^{(h)} \cdot S^{(h)})|.$$

Substitute this value in (9.3); then

$$(9.5)\qquad |A|^{(n-h-k-1)C(n-h,k)} E = |A|^{(n-h-1)C(n-h-1,k)} \, |A(R^{(h)} \cdot S^{(h)})|^{C(n-h-1,k)}.$$

Since $(n-h-1)C(n-h-1,\ k)-(n-h-k-1)C(n-h,\ k)=C(n-h-1,\ k-1)$, the relation (9.1) follows from (9.5). The proof is complete.

10. History and the literature. According to David Eugene Smith [41, p. 273], the Chinese had some idea of the determinant as early as about 1300 A.D. The distinguished Japanese mathematician Seki Shinsuke Kōwa, or Takakazu, (1642–1708), discovered the expansion of a determinant [41, pp. 439–440] in solving simultaneous equations. In the West, the earliest use of the functions we call determinants occurs in correspondence between Leibniz and De L'Hospital. In a letter to the latter dated 28th April 1693 Leibniz explains his methods and results. This correspondence was not published until 1850, and Leibniz's discoveries apparently had no influence on later developments.

The actual development of the theory of determinants dates from the publication of a book by Cramer [15, pp. 59–60, 656–659] in Geneva in 1750. In it we find the familiar Cramer's Rule. Vandermonde laid the first real foundation for the theory in an important paper [47] read January 12, 1771. Laplace [26]

advanced the theory. Gauss [18] used determinants and the term "determinant of a form." Papers by Binet [7] and Cauchy [12], both read to the Institute on November 30, 1812, make the year 1812 one of the most important in the history of determinants. Binet's paper contained important results, but Cauchy gave a masterly treatment of the entire subject. Cauchy, borrowing the word from Gauss, used the term determinant in the same sense in which it is used today. Schweins [38] published many important discoveries in a book in 1825. They were lost, however, and the fact that he had ever written on determinants was not brought to light until 1884. The year 1841 was also one of outstanding importance for determinants. Jacobi, who had already published several papers on the subject, brought together all the known results and added to them [21] and, following Cauchy, fixed the use of the name determinant. In a second paper in 1841 Jacobi treated Jacobians [22]. Finally, in 1841, Cayley in his first published paper [13] introduced the present-day standard notation of vertical bars enclosing a square array to denote a determinant. Many notations had been used before this time, but Cayley's introduction of a notation capable of representing all determinants gave a great impetus to the study of the subject.

Let us turn now to the history of the theorems treated in this exposition. Vandermonde [47] gave the Laplace expansion for a determinant of order $2m$ in terms of minors of order 2 and $(2m-2)$. Laplace [26] gave the general theorem, but Cauchy in 1812 gave the first adequate proof of Laplace's expansion in his powerful treatment [12] of the whole subject of determinants.

Lagrange [25, *Oeuvres*, p. 580] gave a lemma which apparently contains the first known example of the multiplication theorem. Lagrange evaluated the square of a determinant of the third order, but it is not at all certain that he recognized the result as a determinant. Gauss [18, Chap. V] definitely used the multiplication theorem (4.2) in connection with the transformation of ternary quadratic forms. Binet [7] published for the first time the multiplication theorem (4.3). Muir [29, vol. I, p. 109] thinks that Cauchy [12] was the first to publish a satisfactory proof of the multiplication theorem, but that Binet gave the more general form of it. Sylvester [46] gave the multiplication theorem in the general form (4.6). Finally, it may be remarked that Sylvester [45] (the result can be found also in [24, pp. 242–243] and [31, p. 223]) gave the multiplication theorem in a completely different form for the product of determinants each of which has a column of 1's; he used this result to prove the theorem in Cayley's first published paper [13].

The first known case of Sylvester's theorem of 1839 and 1851 (see (5.1) above) was given by Fontaine [5] in 1748, before the founding of the theory of determinants. Bézout [6, pp. 171–187, 208–223] was the next to give examples of these identities, but he considered only determinants of the second and third order. Following Bézout, many others (see Muir [29]) gave instances of the identities (5.1), but Sylvester was the first to give the general theorem and its proof. In [42] Sylvester proved (5.1) for the case $k=1$, and in [44] he gave the

completely general theorem and its proof. Bruno [11] gave a more satisfactory proof. A proof of (5.1) can be obtained by using Laplace's expansion on a properly formed determinant [31, pp. 117–122], but this method is not elegant. Beaver [4] has given a proof of the theorem and certain extensions.

Sylvester's claim to the Sylvester-Franke theorem (6.1) rests on the fact that he published [43] in March, 1851 the general theorem (9.1) that includes (6.1) as a special case. He did not call attention to the special case, however, and he gave no proof of the general theorem. The proof given in section 6 above is Franke's proof [17]. Franke's paper has two famous footnotes added by the editor, Borchardt. In the second footnote Borchardt deduces the Sylvester-Franke theorem (6.1) from $|A^{(k)}||\operatorname{adj}^{(k)} A| = |A|^{C(n,k)}$ (see (3.3) above) and the fact (see [1, pp. 37–38]) that a determinant, considered as a polynomial in its elements, is prime (for this proof, see [1, p. 92] or [31, pp. 178–179]). It is rather remarkable that, for a result as fundamental as the Sylvester-Franke theorem, these two proofs are the only direct ones found in the literature. Other proofs, by no means simple or direct, are known; see for example the one obtained incidentally by Stéphanos [49, pp. 108–110 (there appears to be a consistent error in the exponent in the statement of the theorem)].

Bazin [3] was the first to treat the Bazin-Reiss-Picquet theorem (7.1) (he considered only the case $k=1$). The general theorem (7.1) was given by Reiss [36]. Picquet rediscovered numerous related theorems [34] and gave a comprehensive exposition of the entire subject [35]. Rubini [37] and Hamburger [19] showed that Sylvester's theorem of 1839 and 1851, in the case $k=1$, can be deduced from the Bazin-Reiss-Picquet theorem in the case $k=1$. Zehfuss [48] showed that Jacobi's theorem (8.6) can be deduced from the Bazin-Reiss-Picquet theorem in the case $k=1$. Picquet [35] showed that Sylvester's theorem on superdeterminants (9.1) can be deduced from the Bazin-Reiss-Picquet theorem, but the proof is not very elegant in the general case (this proof in the case $k=1$ is given in section 7 above).

Jacobi's theorem (8.6) was published for the first time by Jacobi [20]. Franke [17] proved Franke's theorem (8.7), and the editor, Borchardt, gave a simpler proof in the first of the famous footnotes. Reiss [36] gave Reiss' theorem.

Sylvester [43] stated Sylvester's theorem on superdeterminants (9.1) in March, 1851, but he gave no proof. Jacobi [22] had used the identity in the special case $k=1$ in connection with Jacobians (see [29, vol. I, pp. 381–385; vol. II, p. 192]). In 1854 Brioschi deduced the special case of Sylvester's theorem from Jacobi's identity (see [29, vol. II, p. 250]). The general theorem remained without proof until Reiss [36] supplied one in 1867. Later, proofs were given by Picquet [35], Scott [39], and others (several proofs are found in [24, pp. 76–80, 90–94, 100–101]).

One might hope to discover certain general methods for establishing identities in the theory of determinants, and some progress has been made in this direction. Muir [28] gave two general principles which he called the "Law of Complementaries" and the "Law of Extensible Minors" (see also [31, pp. 172–

173, 179–182] and [32]). Nanson [32] has given an impressive list of applications of these two principles. For example, the Sylvester theorem on superdeterminants (9.1) becomes almost trivial as the extensional of the Sylvester-Franke theorem (6.1). The latter theorem, however, remains without a simple, direct proof.

Aitken's book [1] contains a good modern treatment of determinants, but it is by no means complete. Bôcher [9] gives a good introduction but only a few of the theorems treated in this paper. Cullis' two volumes [16] (see the review by Shaw [40]) contain a wealth of material, but they form nevertheless a highly unsatisfactory work on determinants. Kowalewski's book [24] has long been a standard reference work on determinants, but it has certain faults (see the review by Bôcher [10]) and it is not complete. For example, Kowalewski does not properly explain and exploit the relationship of determinants to matrices; furthermore, he does not give the theorems in sections 5 and 7 above. The treatise by Muir-Metzler [31] contains a great deal of material, much of it not readily accessible elsewhere, but it nevertheless leaves much to be desired. Pascal's book [33] provides a valuable reference work which includes all the material of this paper and supplies complete references to the original sources. Turnbull [50] treats, among other things, applications of the identities of this paper in the important theory of symbolic invariants.

Sir Thomas Muir, the historian of the theory of determinants, has assembled the record up to 1920 in five volumes [29, 30]. He gives reviews of all papers and books on determinants, and the complete history of the theorems treated in this exposition can be traced in his five volumes.

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INDUSTRIAL EXPERIENCE FOR MATHEMATICS PROFESSORS

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1. Introduction. During the war a great number of professors of mathematics assumed positions in industry and governmental agencies; there they applied their training, knowledge, and experience to the technical problems of modern warfare. The necessities of war made "pure" mathematicians turn to applied mathematics, and caused professors to leave their academic life and be engulfed by the demands of industry. At the conclusion of the war most of these professors returned to the college campus to resume their teaching and research. Have they benefited by their experience? What are the effects of the adventure? Will the universities derive benefits from the experiences gained by their faculty members while they were on leave of absence? Has industry benefited by the presence of these highly skilled scientists?

These are questions which we may well pause to consider and thus hope to obtain the answer to the important question: *Should teachers of mathematics participate in the work of industry and governmental agencies during times of peace?*

To arrive at a consensus of opinion on these questions would necessitate an extensive survey. No one man can give the complete answer. However, some evidence may be derived from personal experience, and it should be worth while to publicize the experience and exhibit some obvious facts from which the reader may draw his own conclusions.

It is the contention of the writer that the answers to the above questions are all *yes* and that much can be derived from having members of a college staff spend some time with an industrial organization. The benefits are equally divided among the individual, the university, and industry. It is his belief that temporary positions in industry should be sought by the younger members of any college staff and should be made available by industry. It is the purpose of this paper to expound this idea by drawing from the author's personal experience and citing examples of existing plans for the temporary employment of college teachers. In particular, the aim is to arouse mathematicians to the existence of, the need for, and the participation in such employment.

2. Means of temporary employment. The importance of advanced mathematics to industry was realized by many industrial organizations (noticeably the aircraft, petroleum, and electrical industries†) prior to the war. These organizations had for some time been employing mathematicians as integral members of their research staffs. The employment was both full time and part time. Thus a large eastern aircraft corporation devised a plan by which college faculty members were employed in a temporary capacity. This plan has already been discussed for physics teachers by D. C. Martin [2]. Since it may not have received the attention of mathematicians and since it was the writer's pleasure

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† This paper is not concerned with the type of mathematics used in industry. For a discussion of industrial mathematics see [1].

to be employed under this plan during the summer of 1942, it may be advisable to explain it in some detail.

The plan was to employ a group of college professors and instructors for the summer months, that is, the months during which normally the universities were not operating a full schedule. The group was composed of physicists, mathematicians, chemists, and engineers. No particular attention seemed to have been paid to any specific number of men from each field.

The usual procedure was to assign each man a specific problem for which he was qualified and in which he expressed an interest. The corporation seemed to have a large number of problems from which to choose, and the assignment was made after a conference with members of the permanent engineering staff.

After the problem had been assigned the college instructor was referred to a supervisor, one of the permanent engineers who was familiar with the problem. He was then comfortably located at a desk usually in the group headed by his supervisor or in a group working on related problems. All knowledge and information that the organization had on his particular problem was made available to him, and any physical equipment he needed to carry out his investigations were placed at his disposal. If his problem was an immediate production or design problem he was included in all meetings, and was placed on the mailing list for progress reports. If his problem was of the nature of an independent investigation he was left on his own to carry out his work. A rather good scientific library was located in the same building.

At the end of the summer or whenever his investigations had led to a definite result he was expected to submit a complete report on the progress made. Individual conferences or meetings were instigated at his own request.

Thus he was left to carry on his investigations in a congenial and pleasant atmosphere. He had ample association, both professional and social, with other college instructors and also the permanent engineering staff of the corporation. Depending upon his own personality he was from time to time approached by the junior engineers who requested aid on problems of both an advanced and elementary nature within his field. In reality he was an integral part of the organization while he was there and received excellent treatment by the corporation.

This was indeed a workable plan. The number of participants was steadily increasing (until the war made an interruption), and the corporation seemed anxious to continue it. The writer was employed in a similar plan with a smaller industrial concern in Detroit during the summer of 1943. Undoubtedly there exist other examples with which the author is not familiar. That these plans will be reinstated during the future peace years seems fairly certain.

During and since the end of the war there has been created a new source of temporary employment on practical problems, which in many instances take the form of consultant work for mathematicians. The reference is to the government sponsored research groups established by contract in government laboratories and on university campuses. In the years to come this should prove to be an

important opportunity for part time employment in applied mathematics. Contracts established at universities eliminate the necessity of traveling, which is particularly advantageous to men with families.

3. Need for participation and benefits obtained by mathematicians. R. Weller has stated:

"The writer wishes also to comment on the desirability of the interchange of personnel between government and other activities. Such interchange between university faculties has become well accepted and the practice of encouraging faculty members to associate themselves with industry during vacation periods is common, for in such interchange lies a major opportunity for rapid professional development. Some government agencies have initiated procedures whereby summer work for college students is possible with a view to later full time employment. This might well be extended to include faculty members who could be given leave of absence for the purpose, and could also be encouraged with profit by the larger industrial organizations. If both government and industry (and why not universities?) are well informed regarding each other's activities and a common ground of experience is established, much can be done to promote understanding and cooperation" [3].

Although this statement is primarily directed toward engineers it is equally applicable to mathematicians. There seem to be many reasons why mathematicians should participate in temporary industrial positions. They may be summarized in the following statements:

The added experience should make for a better individual.

Mathematicians have an obligation to fulfill to industry.

It behooves mathematicians to keep abreast with industrial developments in order to preserve the scientific position of mathematics.

Elaborations on these statements may be presented in volumes of words. However, it is more important to be precise and to the point in order that the essentials may not be lost in a maze of verbal eloquence. Suffice it therefore to list the following very essential points.

(1) The emphasis on applied mathematics brought about by the war will carry over in our postwar needs. It has become quite apparent that we need to train more and more applied mathematicians who will fit into industry and governmental organizations as professional mathematicians. In order to train these men more efficiently, members of a university staff must be familiar with the mathematical problems of industry.

(2) A constant complaint by our students and by young men in industry is that professors lack the necessary experience from which to draw practical situations for their illustrations. The complaints are entirely too numerous to be unfounded, and we can not continue to ignore them.

(3) It has also been a constant source of argument, that college curricula in mathematics are designed principally for the training of teachers of mathematics and not industrial mathematicians, physicists, or engineers. If present curricula

are changed, they should be changed by men who have had experience in both teaching and industry.

(4) The quality of mathematics teaching can always be improved; the added experience should make for better mathematics teachers.

(5) Research problems in applied mathematics are frequently buried in industry. University professors do not know of the problems and thus do not contribute to the necessary research. This sad situation should be corrected.

The benefits derived by the participating individual are numerous. They were adequately summarized for the physicist by Dr. Martin **[2]**, and apply also to the mathematician:

"Opportunities for teacher:

(1) To learn, at first hand, the types of problems facing industry, the practical applications of physics to their solution, and the limitations, or approximations necessary, in applying physical principles to production methods.

(2) To discover weaknesses in present methods of training physicists and engineers, and thus to improve their own curricula and methods of teaching.

(3) To work with young physicists and engineers, to learn their viewpoints and share experiences with them.

(4) To meet and share experiences with teachers from other institutions.

(5) To learn the uses of physical apparatus in making industrial measurements and tests.

(6) To observe and perform research, which in turn might lead to an interest in some specific research problem that might be continued at the teacher's own institution.

(7) To travel and live in a different section of the country for a few months.

(8) To provide an income for the summer in a congenial and interesting type of work.

(9) To obtain a permanent position with an industrial organization, if a teacher eventually decided that he was better qualified for, and preferred, that type of work."

4. The need for participation by industry and government agencies. Primarily, the reason why industry would undertake the employment of college professors would be for the applicable results of their efforts, the so called "dollar value." This of course is the obvious reason and needs no further elaboration. However, there exists some far reaching and underlying results on which we may well dwell. Again we quote Dr. Martin **[2]**:

"Some advantages to an industrial organization:

(1) A better understanding of, and a more sympathetic attitude towards, the technical and economic problems of industry in general, and of one industry in particular, on the part of the teachers.

(2) Association of their young engineers and physicists with men of higher academic training and wider experience; this should help to remind the

younger men of the importance of continual study and reading in their particular field.

(3) Provide in the future better trained engineers and industrial physicists, as their teachers would be more familiar with the problems of industry.

(4) Provide additional trained men to work on engineering and research problems; short problems could be devised or longer ones initiated, which might be continued at the teacher's own institution.

(5) Provide, incidentally, good advertising for the products of a particular industry."

These are points which are indeed well worth considering and which apply equally well to mathematicians. However, we should like to go further by adding a few more points which are deemed to be pertinent.

The employment of college professors would provide any industrial organization with a personal contact within one or more universities. This contact could result in obtaining recommendations for graduates seeking positions in the particular organization; recommendations that would really mean something since the recommender is now familiar with the organization as well as the prospective employee. This personal contact could result in the "farming out" of long term research programs, research which could not be expected to be borne by any one organization but which, nevertheless, would benefit that organization. There are other ways in which "personal contact" could prove advantageous.

There is one objection that has been raised by industry with which the writer takes exception; namely, that it takes a man at least six months or more in industry before he is of any practical value to the organization, and therefore short time employment is not beneficial. The mere fact that industry hires consultants disproves this statement. Furthermore, the member of a college staff hired is already a highly trained individual in his own field and if his work is limited to his field or neighboring problems, the results of his investigations will prove fruitful.

The plan presented here should be "a gift from the gods" to the small industries who cannot afford to hire permanent specialized scientists such as mathematicians, physicists, chemists, or the like.

5. Conclusion. Throughout this paper it has been pointed out that mathematicians and industry would mutually benefit from temporary employment of college mathematics professors in industry. That the university would benefit should be very obvious; if for no other reason, the improved teaching which will result is indeed ample reward to the university. Furthermore, the university has an obligation to industry which in part can be met by making teaching personnel available to industry. Finally, we again point to Weller's statement [3], "If both government and industry (and the university) are well informed regarding each other's activities and a common ground of experience is established much can be done to promote understanding and cooperation."

It is the belief of the writer that universities should recognize the worth of this part-time employment of their faculty members by

(1) obtaining contracts for research to be conducted at the university,

(2) encouraging their faculty members to assume temporary positions in industry, and

(3) recognizing the merits of the plan by suitable promotions based on such employment.

It is hoped from this article that mathematicians now engaged in college teaching will derive some enthusiasm for participating in the solution of industrial problems by obtaining industrial experience, and that mathematicians will look upon a temporary position in industry as an opportunity to become better mathematicians and teachers. If this is too high an aim, it is hoped that mathematicians will at least take cognizance of some of the viewpoints exhibited above.

References

1. T. C. Fry, Industrial Mathematics, this MONTHLY, vol. 48, No. 6, Part II, Supplement, June-July, 1941.

2. D. C. Martin, Industrial Experience for College Teachers, American Journal of Physics, vol. 13, No. 1, Feb., 1945, p. 46.

3. R. Weller, Federal Government Employment For Scientific and Technical Workers, Engineering Experiment Station News, The Ohio State University, vol. XVII, No. 5, Dec., 1945, p. 38.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

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ON COMPLETING A DETERMINANT

V. L. KLEE, JR., University of Virginia

As is well known, to each pair of integers a_1, a_2, with greatest common divisor $(a_1, a_2) = d$, there corresponds a pair of integers k_1, k_2, such that

$$k_1a_1 + k_2a_2 = d.$$

Indeed, by repeated applications of this fact, it is easily shown that if

$$(a_1, a_2, \cdots, a_n) = d,$$

then there are integers $k_1, k_2 \cdots, k_n$, for which

$$k_1a_1 + k_2a_2 + \cdots + k_na_n = d.$$

Now the fact mentioned above, for $n = 2$, can also be stated thus: Given $(a_1, a_2) = d$, we can find a_1' and a_2' so that the determinant

$$\begin{vmatrix} a_1 & a_2 \\ a_1' & a_2' \end{vmatrix}$$

has value d. Generalizing in this direction, we state the following theorem:

THEOREM: *If* $(a_{11}, \cdots, a_{1n}) = d$, *there are integers* $a_{ji} (j = 2, \cdots, n;\ i = 1, \cdots, n)$ *such that the determinant* $|a_{ji}|$ *has value* d.

To prove this theorem it clearly will suffice to consider only the case $d = 1$. We will use the following theorem from linear algebra:*

If U *is a non-degenerate subgroup of the* n*-membered modulus** M, *then there is a basis* $y_1, \cdots, y_n$ *of* M *and there are positive integers* $k_1, \cdots, k_r$, *for some* $r \leqq n$, *such that* $k_1 y_1, \cdots, k_r y_r$ *is a basis for* U.

Now suppose $(a_{11}, \cdots, a_{1n}) = 1$, and let $x_1, \cdots x_n$ be an arbitrary basis for M. Let U consist of all elements in M of the form $k\sum_1^n a_{1i}x_i$, where k is an integer. Then since U is a non-degenerate subgroup of M, we can pick a basis $y_1, \cdots, y_n$ of M having the properties stated in the theorem. But clearly some element y_i must be $\pm\sum_1^n a_{1i}x_i$, and we may assume our indices chosen so that it is y_1. Now for $j = 2, \cdots, n$, we can write $y_j = \pm\sum_1^n a_{ji}x_i$. Then, since $y_1, \cdots, y_n$ is a basis for M, we have, from another well-known theorem, $|a_{ji}| = \pm 1$. Thus, by changing the signs in one row if necessary, we have proved the theorem.

A PAIR OF RECURSION RELATIONS

S. T. PARKER, University of Cincinnati

1. Introduction. Problem E 695 of this MONTHLY, vol. 52, 1945, p. 516, proposed by H. L. Lee, asked for triangles whose sides are integers in arithmetic progression, and whose areas are integers. The problem involved finding integral solutions of the equation

$$x^2 = 3y^2 + d^2.$$

In solving this equation for $d = 1$, the author used certain recursion relations, and then adapted them to the solution of the well-known Pellian equation†

$$x^2 = cy^2 + 1, \qquad 0 < c \neq z^2. \tag{1}$$

In the work that follows, (a, b) will represent a solution in integers of (1); thus

$$a^2 = cb^2 + 1. \tag{2}$$

The chief results of this paper are a theorem which we have called Lemma 4, and the following theorem:

THEOREM: *If the number pair* (a, b) *is such that* b *is the smallest (positive) integer satisfying* (2), *and if* $x_0 = 1$, $x_1 = a$, $y_0 = 0$, $y_1 = b$, *then the two recursion relations*

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* See, for example, Alexandroff and Hopf, Topologie I. By 'modulus' is meant a linear-form modulus with integers as multipliers.

† See Perron, Die Lehre von den Kettenbrüchen, p. 102.

$$\begin{cases} x_n = ax_{n-1} + bcy_{n-1} \\ y_n = bx_{n-1} + ay_{n-1} \end{cases} \qquad n \geqq 1, \tag{3}$$

and

$$\begin{cases} x_n = 2ax_{n-1} - x_{n-2} \\ y_n = 2ay_{n-1} - y_{n-2}, \end{cases} \qquad n \geqq 2, \tag{4}$$

are equivalent.

2. Lemmas. We need several lemmas.

LEMMA 1. *The recursion relations* (3) *give solutions of* (1).

Proof: This is easily proved by mathematical induction.

LEMMA 2. *The recursion relations* (4) *give solutions of* (1).

Proof: Applying the method of mathematical induction to prove that

$$x_n^2 - cy_n^2 = 1, \tag{5}$$

we are led to the necessity of the relation

$$x_n x_{n-1} - cy_n y_{n-1} = a. \tag{6}$$

This auxiliary condition is satisfied for $n=1$. Assume that (5) and (6) hold for $n=1, 2, \cdots, k-1$. We show first that

$$x_k^2 - cy_k^2 = 1,$$

and then that

$$x_k x_{k-1} - cy_k y_{k-1} = a,$$

and the conclusion follows immediately.

LEMMA 3. *The set of numbers given by* (4) *contains the set given by* (3).

Proof: Rewrite (3) in the form

$$\begin{cases} x_{n-1} = ax_{n-2} + bcy_{n-2} \\ y_{n-1} = bx_{n-2} + ay_{n-2}. \end{cases} \tag{7}$$

From (7), using (2), we obtain

$$bcy_{n-1} = ax_{n-1} - x_{n-2},$$

$$bx_{n-1} = ay_{n-1} - y_{n-2}.$$

Relations (4) follow on substituting these results in (3).

LEMMA 4.* *If* (a, b) *is the pair described in the theorem, then the relations* (3) *give all the solutions of* (1).

* I am indebted to Dean W. H. Durfee of Hobart College for the method of proof used in this lemma.

Proof: Assume the lemma false. Then there exist pairs of integers (x, y), satisfying (1) and not obtainable from (3). Therefore there must be one pair (x_m, y_m), from this set, for which y_m is the least. Since $(x_0=1, y_0=0)$ occurs in the set (3), $y_m>b$.

Then

$$x_m^2 = cy_m^2 + 1 = y_m^2\left[c + \frac{1}{y_m^2}\right] < y_m^2\left[c + \frac{1}{b^2}\right] = y_m^2 \cdot \frac{a^2}{b^2}.$$

Therefore

$$x_m < \frac{a}{b}\, y_m.$$

Suppose that

$$x_m \leqq \frac{a-1}{b}\, y_m.$$

This would yield

$$x_m^2 - cy_m^2 \leqq \left\{\frac{(a-1)^2}{b^2} - c\right\} y_m^2 = \frac{2-2a}{b^2}\, y_m^2 < 0,$$

since $c>1$ and $b\geqq 1$, and therefore $a>1$.

This contradiction leads to the double inequality

$$\frac{a-1}{b}\, y_m < x_m < \frac{a}{b}\, y_m. \tag{8}$$

The formulas inverse to (3) are easily found to be

$$\begin{cases} x_{n-1} = ax_n - bcy_n \\ y_{n-1} = -bx_n + ay_n. \end{cases} \tag{9}$$

On replacing (x_n, y_n) in (9) by (x_m, y_m), we obtain a new pair (x_{m-1}, y_{m-1}) which is found to satisfy (1). Moreover, from (8) and (9), we have

$$y_{m-1} > -b\left(\frac{a}{b}\right) y_m + ay_m = 0,$$

$$y_{m-1} < -b\left(\frac{a-1}{b}\right) y_m + ay_m = y_m.$$

Thus

$$y_m > y_{m-1} > 0,$$

and we have found a pair (x_{m-1}, y_{m-1}) with a positive y_{m-1} less than y_m. If (x_{m-1}, y_{m-1}) is a pair given by relations (3), so is (x_m, y_m), by applying (3) to (x_{m-1}, y_{m-1}). Therefore (x_{m-1}, y_{m-1}) cannot be in our set, and the contradiction proves the lemma.

3. Proof of the Theorem. We note that the number pairs given by (3) are also given by (4), from Lemma 3. The inclusion of (4) in (3) follows from considering Lemmas 4 and 2. Thus the two sets of numbers are the same.

4. Remarks. (1) An interesting incidental result is that the number pairs given by (3), or (4), also satisfy the relation

$$x_n x_{n-p} = c y_n y_{n-p} + x_p, \qquad n \geqq p \geqq 0.$$

(2) It would seem that the ratio a/b, of the numbers a, b described in the theorem, is the first convergent p/q of the continued fraction expansion for $\sqrt{c}$ for which $p^2 = cq^2 + 1$. This p/q is the penultimate* convergent of the first or second period, according as the period is even or odd. (If the period is one, p/q is the second convergent.)

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania. Contributions are invited on topics of immediate interest to teachers of undergraduate mathematics such as: fresh approaches to standard material, analyses of common textbook shortcomings, descriptions of visual and mechanical aids to teaching, outlines of new types of courses, and discussions of the role of mathematics in the revised curricula being adopted by many institutions. Rejoinders to earlier notes are encouraged.

THE REDUCED EQUATION OF THE GENERAL CONIC

A. E. JOHNS, McMaster University

EDITOR'S NOTE. The subject matter of this note is familiar to everyone, and has received a satisfactory treatment in a number of standard texts. Among such expositions, that of Professor Johns is outstanding for its elegance and precision, and it is being published for this reason.

1. Notation. The general equation of a conic may be written:

$$aX^2 + 2hXY + bY^2 + 2gX + 2fY + c = 0.$$

Let

$$D \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

and let A, B, C, F, G, H be the cofactors of the corresponding small letters in relation D. The fundamental invariants under rotation and translation of rectangular axes are: $a+b$; $C = ab - h^2$; and D. In addition, when $C=0$ and $D=0$, $A+B = bc+ca-f^2-g^2$ is also invariant. The discriminating quadratic is:

$$\lambda^2 - (a+b)\lambda + ab - h^2 = 0$$

* See Hall and Knight, Higher Algebra, Ch. XXVIII.

with roots λ_1 and λ_2.

2. Rotation of axes. The equations of transformation for a rotation through an angle θ are:

$$X = lx - my; \qquad Y = mx + ly$$

where $l = \cos\theta$ and $m = \sin\theta$; and the new equation of the conic takes the form:

$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$$

where:

$$a' = al^2 + 2hlm + bm^2 \qquad f' = fl - gm$$
$$b' = am^2 - 2hlm + bl^2 \qquad g' = gl + fm$$
$$c' = c \qquad h' = (b - a)lm + h(l^2 - m^2).$$

It is always possible to choose a positive acute angle θ to satisfy the equation: $\tan 2\theta = 2h/(a-b)$ and so make $h' = 0$. For this θ, l and m are both positive. After this rotation through the positive acute angle θ, and a suitable translation if necessary, the equation of the conic takes on one of the following types of reduced forms.

Case 1. When $C \neq 0$, $\lambda_1 x^2 + \lambda_2 y^2 + D/C = 0$.

These are the central conics, including two intersecting lines.

Case 2. When $C = 0$ and $D \neq 0$,

(i) $$\lambda_1 x^2 + 2f'y = 0, \quad \text{where} \quad f' = \pm\left(\frac{-D}{\lambda_1}\right)^{1/2},$$

or

(ii) $$\lambda_2 y^2 + 2g'x = 0, \quad \text{where} \quad g' = \pm\left(\frac{-D}{\lambda_2}\right)^{1/2}.$$

These are true parabolas.

Case 3. When $C = 0$ and $D = 0$,

(i) $$\lambda_1 x^2 + \frac{A + B}{\lambda_1} = 0,$$

or

(ii) $$\lambda_2 y^2 + \frac{A + B}{\lambda_2} = 0.$$

These are parallel lines.

The above forms follow directly from the fact that the expressions listed in §1 are invariant.

3. Discussion. When a student asks the instructor which root of the discriminating quadratic is λ_1, or which of the alternative forms in Cases 2 and 3 is correct, or which sign for f' and g' should be chosen, the answer sometimes given is that it does not matter and that either is correct depending on how the new axes are named. However, if θ is to be a positive acute angle this reply is not correct. The following is an attempt to determine the proper alternatives on the above assumption.

From the equations of §2 above we have when $h'=0$

$$\lambda_1 - \lambda_2 = a' - b' = (a - b)(l^2 - m^2) + 4hlm,$$

and

$$h' = (b - a)lm + h(l^2 - m^2) = 0.$$

So

$$\lambda_1 - \lambda_2 = \left\{\frac{(a-b)^2}{h} + 4h\right\} lm = \left\{(a-b)^2 + 4h^2\right\} \frac{lm}{h}.$$

Since lm and $(a-b)^2+4h^2$ are always positive, λ_1 and λ_2 must be so chosen that $\lambda_1-\lambda_2$ has the same sign as h.

For the central conics of Case 1 where $C\neq 0$, this rule is sufficient to determine which root is λ_1, and which is λ_2 in the reduced form

$$\lambda_1 x^2 + \lambda_2 y^2 + \frac{D}{C} = 0.$$

For Cases 2 and 3 where $C=0=ab-h^2$, a and b must always have the same sign. For simplicity, let us suppose that our original equation is written with these coefficients both positive. Then $a+b$ is also positive. When $C=0$, one root of the discriminating quadratic is zero and the other is $a+b$. Two possibilities now arise depending on the sign of h. If h is positive, $\lambda_1=a+b$ and $\lambda_2=0$, since $\lambda_1-\lambda_2$ must have the same sign as h. So the reduced forms are, respectively,

$$\lambda_1 x^2 + 2f'y = 0 \quad \text{and} \quad \lambda_1 x^2 + \frac{A+B}{\lambda_1} = 0.$$

If h is negative, $\lambda_1=0$ and $\lambda_2=a+b$, since $\lambda_1-\lambda_2$ must now be negative to match h in sign. The reduced forms are now, respectively,

$$\lambda_2 y^2 + 2g'x = 0 \quad \text{and} \quad \lambda_2 y^2 + \frac{A+B}{\lambda_2} = 0.$$

It remains to determine the signs of f' and g' on the understanding that θ is a positive acute angle.

When $ab=h^2$ and $h'=hl^2+(b-a)lm-hm^2=0$, we have

$$h^2l^2 + (b - a)hlm - abm^2 = 0$$

or

$$(hl - am)(hl + bm) = 0,$$

whence

$$\frac{l}{a} = \frac{m}{h} \quad \text{or} \quad \frac{l}{b} = \frac{m}{-h}.$$

When h is positive,

$$\frac{l}{a} = \frac{m}{h} = \frac{1}{(a^2 + h^2)^{\frac{1}{2}}} = \frac{1}{(a^2 + ab)^{\frac{1}{2}}}.$$

From §2

$$f' = fl - gm = \frac{af - gh}{(a^2 + ab)^{\frac{1}{2}}}.$$

This form, which is equivalent to $\pm\left(\frac{-D}{a+b}\right)^{\frac{1}{2}}$ with the proper sign, shows that the sign of f' is the same as the sign of $af-gh$, or opposite to that of F, the cofactor of f in D.

When h is negative,

$$\frac{l}{b} = \frac{m}{-h} = \frac{1}{(b^2 + h^2)^{\frac{1}{2}}} = \frac{1}{(ab + b^2)^{\frac{1}{2}}}.$$

From §2

$$g' = gl + fm = \frac{bg - fh}{(ab + b^2)^{\frac{1}{2}}}.$$

This form, which is equivalent to $\pm\left(\frac{-D}{a+b}\right)^{\frac{1}{2}}$ with the proper sign, shows that the sign of g' is the same as the sign of $bg-fh$, or opposite to that of G, the cofactor of g in D.

4. Examples:

EXAMPLE 1. $8x^2-4xy+5y^2-36x+18y+9=0$.

$a = 8 \quad f = 9 \quad a + b = 13$

$b = 5 \quad g = -18 \quad C = ab - h^2 = 36$

$c = 9 \quad h = -2 \quad D \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = -36^2.$

$\tan 2\theta = -4/3$, $\tan \theta = 2$. Choose θ a positive acute angle.

$$\lambda^2 - 13\lambda + 36 = 0 \quad \text{or} \quad (\lambda - 4)(\lambda - 9) = 0.$$

Since h is negative, choose $\lambda_1 = 4$, $\lambda_2 = 9$, to make $\lambda_1 - \lambda_2$ negative. The reduced equation $\lambda_1 x^2 + \lambda_2 y^2 + D/C = 0$ becomes $4x^2 + 9y^2 - 36 = 0$.

EXAMPLE 2. $x^2-4xy+4y^2+5y-6=0$.

$$a=1 \qquad f=\frac{5}{2} \qquad a+b=5$$

$$b=4 \qquad g=0 \qquad ab-h^2=0$$

$$c=-6 \qquad h=-2 \qquad D=\frac{-25}{4}$$

$\tan 2\theta=4/3$, $\tan\theta=1/2$, $\lambda^2-5\lambda=0$.
Since h is negative, choose $\lambda_1=0$ and $\lambda_2=5$ to make $\lambda_1-\lambda_2$ negative. $g'=+5^{\frac{1}{2}}/2$ and the positive sign must be chosen since G is negative. The reduced equation is $5y^2+5^{\frac{1}{2}}x=0$.

EXAMPLE 3. $4x^2+12xy+9y^2+2x+3y-42=0$.

$$a=4 \qquad f=\frac{3}{2} \qquad a+b=13$$

$$b=9 \qquad g=1 \qquad ab-h^2=0$$

$$c=-42 \qquad h=6 \qquad D=0 \qquad A+B=-\frac{13^3}{4}$$

$\tan 2\theta=-12/5$, $\tan\theta=3/2$, $\lambda^2-13\lambda=0$.
Since h is positive, $\lambda_1=13$, $\lambda_2=0$ and the reduced form is $13x^2-13^2/4=0$ or $x=\pm 13^{\frac{1}{2}}/2$.

SIMPLIFICATION OF EQUATIONS OF CONICS

M. T. BIRD, Allegheny College

I should like to extend the discussion of the simplification of equations of conics initiated by Thornton* and continued by Johnston.†

Consider the general equation of the conic in the form

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0. \tag{1}$$

It is assumed that B is not zero. The equation of the conic may be written in the alternative form

$$A(x^2+y^2)+y[Bx-(A-C)y]+Dx+Ey+F=0. \tag{2}$$

The angles between the pair of intersecting lines

$$y[Bx-(A-C)y]=0 \tag{3}$$

are bisected by the mutually perpendicular lines

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* Simplification of the equations of conics, this MONTHLY, vol. 41, 1934, p. 36.
† Simplification of equations of conics, this MONTHLY, vol. 44, 1937, p. 30.

$$\pm y = \frac{Bx - (A - C)y}{\sqrt{B^2 + (A - C)^2}}. \tag{4}$$

It follows that the lines (4) are axes of symmetry for the degenerate conic (3). This fact may be used to simplify equation (2).

Let the mutually perpendicular lines (4) be written in the normal form

$$0 = mx + ny,\ 0 = -nx + my;\quad m > 0,\ n \geqq 0,\ m^2 + n^2 = 1. \tag{5}$$

The quantities x', y' defined by the equations,

$$x' = mx + ny,\ y' = -nx + my;\quad m > 0,\ n \geqq 0,\ m^2 + n^2 = 1, \tag{6}$$

measure the distances of the point $P(x, y)$ from the mutually perpendicular lines (5). The quantities x', y' constitute coördinates of the point $P(x, y)$ after a rotation of axes, and the equations (6) define the coördinate transformation whose inverse is seen to be

$$x = mx' - ny',\ y = nx' + my';\quad m > 0,\ n \geqq 0,\ m^2 + n^2 = 1. \tag{7}$$

The symmetry of the degenerate conic (3) about the lines (5) leads to the conclusion that the equation of the degenerate conic (3) will assume the form

$$y[Bx - (A - C)y] \equiv Mx'^2 + Ny'^2 = 0,$$

under the transformation (7). It is readily verified that the expression x^2+y^2 is invariant under the transformation (7). Consequently the transformation (7) carries the general equation of the conic (2) into the form

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0.$$

Consider the example

$$9x^2 - 24xy + 16y^2 - 186x - 252y - 63 = 0.$$

Following the method outlined above we have

$$9(x^2 + y^2) - y(24x - 7y) - 186x - 252y - 63 = 0.$$

The new coördinate axes are defined by the equations

$$\pm y = (24x - 7y)/25,$$

which may be written

$$24x + 18y = 0,\qquad 24x - 32y = 0.$$

The new coördinates are defined by the equations

$$x' = (4x + 3y)/5,\qquad y' = (-3x + 4y)/5.$$

The transformation

$$x = (4x' - 3y')/5,\qquad y = (3x' + 4y')/5$$

leads to the equation of the conic in the form

$$25y'^2 - 300x' - 90y' - 63 = 0.$$

In this example the given equation may be written

$$(3x - 4y)^2 - 186x - 252y - 63 = 0.$$

The parenthetical expression suggests at once the introduction of the coördinates defined above in order to obtain the equation

$$y'^2 + ax' + by' + c = 0.$$

AN APPLICATION OF THE REMAINDER THEOREM

R. W. WAGNER, Oberlin College

The writer of this note has long felt that a student's appreciation of the remainder theorem would be enhanced if more applications of the theorem were presented to him. In this note the remainder theorem will be used to establish two well known rules for the divisibility of one number by another. The author has been unable to find this application in the literature and would be glad to know of a reference to this use of the remainder theorem.

The conclusion of the remainder theorem can be written as the identity

$$\frac{p(x)}{x-a} \equiv q(x) + \frac{p(a)}{x-a}. \tag{1}$$

In the present applications, $p(x)$ is to be a polynomial whose coefficients are in the set of numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Then $p(10)$ is the ordinary decimal notation for an integer. Also, $p(1)$ is the sum of the digits of the integer.

In the identity (1) assign to x the value 10 and to a the value 1. This leads to the equation

$$\frac{p(10)}{9} = q(10) + \frac{p(1)}{9}. \tag{2}$$

The expression $q(10)$ will represent an integer, though perhaps not in its proper decimal form. Now, if $p(1)$ is divisible by 9, $p(10)/9$ is an integer and $p(10)$ is divisible by 9. By transposing $q(10)$, the converse is similarly established. Therefore, *an integer is divisible by* 9 *if, and only if, the sum of its digits is divisble by* 9.

If both members of equation (2) are multiplied by 3, the result is

$$\frac{p(10)}{3} = 3q(10) + \frac{p(1)}{3}.$$

Again, $3q(10)$ is an integer. Hence, *an integer is divisible by* 3 *if, and only if, the sum of its digits is divisible by* 3.

A rule for the divisibility of an integer by 11 can be derived from the identity (1) by assigning to x the value 10 and to a the value -1.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

E 756. *Proposed by G. Pólya, Stanford University*

Show that

$$\begin{vmatrix} a-x & 1 & 0 & 0 & \cdots & 0 \\ \binom{a}{2} & a-x & 1 & 0 & \cdots & 0 \\ \binom{a}{3} & \binom{a}{2} & a-x & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \binom{a}{n} & \binom{a}{n-1} & \binom{a}{n-2} & \binom{a}{n-3} & \cdots & a-x \end{vmatrix}$$

$$= \binom{a+n-1}{n} - \binom{2a+n-2}{n-1}x + \binom{3a+n-3}{n-2}x^2 - \cdots + (-1)^n x^n.$$

E 757. *Proposed by Kirkland Stewart, College of Puget Sound*

A wedge is cut from a right circular cylinder by an oblique plane passing through a diameter of the base of the cylinder. Find the volume of the wedge using Cavalieri's Theorem.

E 758. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A *kite* consists of the area bounded by a major arc of a circle of radius r and the two tangents drawn at the endpoints of the arc. Show that (1) the area of the kite is equal to half the product of its perimeter by the radius r, (2) $Og/OG = 3/2$, where g and G are the centroids of the perimeter and area, and O is the center of the kite's arc, (3) $Og'/OG' = 4/3$, where g' and G' are the centroids of the surface and volume of the solid of revolution obtained by revolving the kite about its axis, (4) the plane through G' perpendicular to the axis bisects the lateral area of the solid, (5) the volume of the solid is equal to one third the product of its surface by the radius r.

E 759. *Proposed by Theodore Running, Ann Arbor, Michigan*

Show that $x^n - (x-a)^n$ can be expressed as the difference of two squares in at least one way for all positive integral values of n and x, a odd and less than x, not counting the obvious way when n is even.

We may let a be even provided n and x are both odd.

E 760. *Proposed by C. D. Olds, San Jose State College*

If

$$u_k = \frac{2 \cdot 6 \cdot 10 \cdots (4k-2)}{2 \cdot 3 \cdot 4 \cdot 5 \cdots (k+1)},$$

find the value of

$$u_n + u_1 u_{n-1} + u_2 u_{n-2} + \cdots + u_{n-1} u_1 + u_n.$$

SOLUTIONS

A Property of a Quadrilateral

E 715 [1946, 157]. *Proposed by L. M. Kelly, University of Missouri*

Suppose $ABCD$ is a proper plane convex quadrilateral and P a point exterior to this plane. Consider the four tetrahedra $PABC$, $PABD$, $PACD$, $PBCD$. If PH is the shortest of all the altitudes of these four tetrahedra, show that H must be interior to $ABCD$. (By a proper quadrilateral is meant one with no three vertices linear.)

Solution by the Proposer. We shall use the following:

LEMMA. Let A and B be two points of a plane q, P a point not on q, and H the foot of the perpendicular from P to q. If C is a point of q such that the half ray HC intersects the trace of AB in say, X, so that C is between H and X, then the perpendicular from C to the plane PAB is shorter than PH.

The proof is easy. We might note that X may be the point at infinity on AB.

THE MAIN THEOREM. Let $ABCD$ be the quadrilateral, and suppose DC intersects AB in M and BC intersects AD in N so that ABM and BCN are the respective orders. This is clearly just a question of notation. Now consider the possible exterior positions of H. If it is on the "exterior side" of AB, then the half ray HB would surely intersect one of the traces AD or DC in a point X such that B lies between H and X, and the distance from B to one of the planes PAD or PDC would be less than PH. Similarly an examination of the "exterior sides" of the other three sides of the quadrilateral reveals a contradiction.

A Poristic Construction

E 713 [1946, 157]. *Proposed by Joseph Rosenbaum, The Milford School, Conn.*

Find a euclidean construction for a non-regular pentagon which has both a circumcircle and an incircle.

Solution by the Proposer. It is known that the condition on two circles that they be respectively the circumcircle and the incircle of a pentagon is

$$(1) \qquad (R^2 - d^2)^3 + 2rR(R^2 - d^2)^2 - 4r^2R^2(R^2 - d^2) - 8r^3Rd^2 = 0,$$

where R, r, d are the two radii and distance between the centers (Jacobi's *Collected Works*, VI, p. 277 *ff*).

Equation (1) has the parametric solution

$$R = 2pq^2,$$
$$r = (q - p)(q + p)^2,$$
$$d = 2p(p^3q + p^2q^2 - pq^3)^{1/2}.$$

Now by taking a pair of otherwise arbitrary values for p and q, with the restriction that they be positive and

$$q > p > (\sqrt{5} - 1)q/2,$$

the quantities R, r, d will be real and positive, and furthermore are constructible by compasses and straight-edge, and by Poncelet's porism any one of the corresponding set of pentagons can be constructed, and because $d \neq 0$, the pentagon will be non-regular.

W. B. Clarke produced an interesting symmetrical non-regular pentagon having an incircle and a circumcircle. To obtain the pentagon inscribe an equilateral triangle ABD in a circle, and then locate C and E on the circle such that $\sphericalangle BAC = \sphericalangle ABE = 40°$. The pentagon $ABCDE$ has an incircle with center at the intersection of the diagonals AC and BE. This pentagon, however, is not constructible with euclidean tools.

Two Applications of Casey's Theorem

E 720 [1946, 220]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

At the vertices of an equilateral triangle three equal circles are drawn externally tangent to the circumcircle. Show that one of the three tangents to these equal circles, from any point whatever on the circumcircle, is equal to the sum of the other two.

E 728 [1946, 333]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

With the vertices A, B, C of an equilateral triangle as centers draw the circles (A), (B), (C) which are concurrent at the center O of the triangle, and then draw an arbitrary circle (D) passing through O. Show that the length of one of the common tangents to the circles (A) and (D), (B) and (D), (C) and (D) is equal to the sum of the lengths of the other two.

Combined solution by Rufus Crane, Ohio Wesleyan University. These two problems are special cases of Casey's Theorem in the geometry of the circle, which states a relation between the lengths of the common tangents to four circles touching a fifth:

$$t_{12}t_{34} \pm t_{13}t_{24} \pm t_{14}t_{23} = 0.$$

In E 720 the four circles are the three equal circles S_1, S_2, S_3 tangent to the circumcircle, and the point on the circumcircle considered as a point-circle S_4. Then $t_{12} = t_{13} = t_{23}$, each being equal to the distance between centers of the equal circles.

In E 728 the four circles are the circles (A), (B), (C), (D), while the fifth circle to which they are tangent is the point-circle O. Here $t_{AB}=t_{AC}=t_{BC}$ as in E 720. Identifying A, B, C, D with 1, 2, 3, 4 we have, in each problem

$$t_{34} \pm t_{24} \pm t_{14} = 0.$$

E 720 was also solved by E. F. Allen, Paul Brock, W. E. Byrne, W. B. Clarke, L. M. Kelly, B. R. Leeds, H. R. Leifer, D. W. Matlock, Norman Miller, C. C. Oursler, C. L. Perry, P. A. Piza, Joseph Rosenbaum, P. D. Thomas, R. H. Urbano, and the proposer.

Most solvers gave a straightforward trigonometrical solution. Byrne employed inversive geometry; Thomas offered a synthetic solution employing the theorems of Stewart and Ptolemy; Urbano used analytic geometry. Several solvers noted that the theorem is also true when the circles are internally tangent to the circumcircle and not greater than the circumcircle. In addition, Rosenbaum generalized to an arbitrary triangle with the conclusion that the product of the tangent to one of the equal circles and the side of the triangle opposite the point of contact of that circle with the circumcircle is equal to the sum of the other two corresponding products. As another generalization he noted that if the radii r_1, r_2, r_3 of the three circles are such that $(R+r_1):(R+r_2):(R+R_3)=a_1^2:a_2^2:a_3^2$, where R is the circumradius and a_i are the sides of the triangle, then the conclusion of E 720 continues to hold.

E 728 was also solved by Paul Brock, L. M. Kelly, P. A. Piza, R. H. Urbano, and the proposer.

It is to be noted that both E 720 and E 728 are generalizations of the classic relation $MA+MB=MC$, where M is any point on the minor arc AB of the circumcircle of the equilateral triangle ABC.

Casey's Theorem may be found in Coolidge, *Treatise on the Circle and the Sphere*, p. 37, Lachlan, *Modern Pure Geometry*, pp. 244–250, or Johnson, *Modern Geometry*, pp. 121–126.

A Property of $x^3 + y^3 = z^3$

E 729 [1946, 333]. *Proposed by F. J. Duarte, Caracas, Venezuela*

Let x, y, z be three real positive numbers such that $x^3+y^3=z^3$, and set

$$\lambda = \sqrt[3]{3(x+y)}/(\sqrt[3]{z-x}+\sqrt[3]{z-y}).$$

Show that

$$(\sqrt[3]{2}+\sqrt[3]{4})/2 < \lambda < \sqrt[3]{3}.$$

Solution by E. P. Starke, Rutgers University. Because of the symmetry of the problem in x and y, we may expect extreme values of λ to occur when $x=0$ (or $y=0$) and when $x=y$. After interchanging x and y if necessary, we may take $x \leqq y < z$. For the cases mentioned we compute easily

$$\lambda = \sqrt[3]{3} \quad \text{and} \quad \lambda = (\sqrt[3]{2}+\sqrt[3]{4})/2,$$

respectively. To show that there are no other extreme values of λ, and hence that the given inequalities must hold (since λ is a continuous function of x, y, z), it seems easiest to substitute

$$x + y = a^3, \qquad z - x = b^3, \qquad z - y = c^3, \qquad a, b, c \text{ positive.}$$

We obtain without difficulty

$$\lambda = \sqrt[3]{3}\, a/(b + c), \qquad a^3 - b^3 - c^3 = 2\sqrt[3]{3}\, abc.$$

By eliminating a between these two equations and substituting $w = c/b$ we find

$$(1 + w)^2(\lambda^3 - 3) = 3w(2\lambda - 3). \tag{1}$$

If we put $d\lambda/dw = 0$, we obtain from (1) the condition for extreme λ as

$$2(1 + w)(\lambda^3 - 3) = 3(2\lambda - 3). \tag{2}$$

Upon eliminating w between (1) and (2) we have

$$(2\lambda - 3)(\lambda^3 - 3)(4\lambda^3 - 6\lambda - 3) = 0.$$

The first factor cannot vanish since it corresponds to $w = -1$. The other factors give $w = 0$ and $w = 1$, whence $x = 0$ and $x = y$, as treated above. Now $x = 0$ is excluded by hypothesis, but $x = y$ is not. Hence λ can actually attain its lower bound, but not its upper, and the conclusion of the theorem should be corrected to read

$$(\sqrt[3]{2} + \sqrt[3]{4})/2 \leqq \lambda < \sqrt[3]{3}.$$

Also solved by Murray Barbour, P. A. Piza, and the proposer.

Since, to five decimal places, $(\sqrt[3]{2}+\sqrt[3]{4})/2 = 1.42366$ and $\sqrt[3]{3} = 1.44225$, we note that λ is quite restricted. The proposer gave the example $x = 10$, $y = \sqrt{82} - 2$, $z = \sqrt{82} + 2$, $\lambda = 1.425$ (approximately).

The Interior Bisectors and the Medial Triangle

E 730 [1946, 333]. *Proposed by J. H. Butchart, Arizona State College*

The interior angle bisectors of a triangle meet the noncorresponding sides of the medial triangle in six points which lie in pairs on the lines joining the points of tangency of the inscribed circle.

Solution by L. M. Kelly, University of Missouri. Consider a triangle ABC, the midpoints of the respective sides being A_1, B_1, C_1. Let the interior bisector of angle A meet A_1C_1 in A_2, and that of B meet B_1C_1 in B_2. Join A_2B_2 and call the intersections of A_2B_2 with AC and BC respectively B_3 and A_3. We propose to show that B_3 and A_3 are points of contact of the inscribed circle. If true, the theorem of the problem would follow at once.

We observe that C_1B_2 is parallel to BC, and hence BC_1B_2 is isosceles. Thus $B_2C_1 = \frac{1}{2}c = A_2C_1$. Furthermore, since B_1B_3 is parallel to A_2C_1, triangle $B_2B_1B_3$ is isosceles, and $B_1B_3 = B_1B_2 = \frac{1}{2}c - \frac{1}{2}a$. Thus $B_3C = \frac{1}{2}(b+c-a) = s-a$, and B_3 is a point of contact of the inscribed circle.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS AND SOLUTIONS

4234. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that if the Monge point of a tetrahedron $ABCD$ lies in the plane of the face BCD, then the altitudes, through vertex A, of the triangles ABC, ACD, ADB are coplanar, and conversely. Examine the cases where the Monge point lies on an edge and at a vertex of the face BCD. Show how to construct tetrahedra illustrating each case.

4235. *Proposed by Irving Kaplansky, University of Chicago, and D. C. Lewis, University of New Hampshire*

Show that the determinant

$$\begin{vmatrix} (x-1)/1 & (x^2-1)/2 & \cdots & (x^n-1)/n \\ (x^2-1)/2 & (x^3-1)/3 & \cdots & (x^{n+1}-1)/(n+1) \\ \cdots & \cdots & \cdots & \cdots \\ (x^n-1)/n & (x^{n+1}-1)/(n+1) & \cdots & (x^{2n-1}-1)/(2n-1) \end{vmatrix}$$

is a constant times $(x-1)^{n^2}$.

4236. *Proposed by H. D. Grossman, New York City*

Write down two 1's, then a 2 between them, then a 3 between any two numbers whose sum is 3, then a 4 between any two numbers whose sum is 4, and so forth. Prove that the number of n's written down is $\phi(n)$, ($n>1$, ϕ is Euler's totient.)

4237. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In which systems of numeration, of base B, are there two-digit squares $m^2=ab$ and $n^2=cd$, if $d=a+1$ and $b=c+1$? Show that, if also $b+d=B+1$, the four-digit numbers $ddbb$ and $bbdd$ are the squares of the numbers mm and nn.

4238. *Proposed by C. E. Springer, University of Oklahoma*

At each point of the revolute

$$x = u\cos v, \qquad y = u\sin v, \qquad z = \phi(u)$$

a straight line generator of a congruence is taken with direction cosines proportional to

$$\cos v, \qquad \sin v, \qquad \phi' - u\phi'',$$

where the primes indicate differentiation with respect to u. Show that the developables of the congruence intersect the revolute in its lines of curvature.

SOLUTIONS

Planes Bisecting the Volume and the Area of a Tetrahedron

3879 [1938, 389]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that it is possible to determine a plane section, limited by three faces of a tetrahedron which divides both the surface and the volume into equal parts. Show that the plane passes through the center of the inscribed sphere.

Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France. Take oblique cartesian space axes Dx, Dy, Dz along the edges DA, DB, DC of the given tetrahedron $DABC$. Let λ, μ, ν denote the angles yDz, zDx, xDy, respectively, and let

$$ux + vy + wz + 1 = 0$$

be the equation of the required plane section. If V denotes the volume and r the inradius of the given tetrahedron, and ω the sine of the solid angle $D-ABC$, we have the relations

$$\omega/uvw = -3V, \tag{1}$$

$$(\sin\lambda)/vw + (\sin\mu)/wu + (\sin\nu)/uv = 3V/r, \tag{2}$$

whence we find

$$(ur\sin\lambda)/\omega + (vr\sin\mu)/\omega + (wr\sin\nu)/\omega + 1 = 0.$$

If DX, DY, DZ are the edges of the trihedral angle supplementary to $D-xyz$, and if α, β, γ denote the angles xDX, yDY, zDZ, respectively, then

$$\omega = \sin\lambda\cos\alpha = \sin\mu\cos\beta = \sin\nu\cos\gamma$$

and we have

$$ur/\cos\alpha + vr/\cos\beta + wr/\cos\gamma + 1 = 0.$$

Since $(r/\cos\alpha, r/\cos\beta, r/\cos\gamma)$ are the coördinates of the incenter of the tetrahedron, this last equation shows that the plane section must pass through the incenter.

There are, then, an infinite number of solutions associated with the vertex D of the tetrahedron. These solutions are given by the tangent planes of a cone having for vertex the incenter of the tetrahedron and circumscribed about the two third class surfaces (1) and (2). Similar infinitudes of solutions are obtained for the other vertices of the tetrahedron.

Editorial Note. In contrast to the fourfold infinitude of solutions for the tetrahedron we find that the analogous problem for the triangle possesses, in general, only six solutions, all passing through the incenter of the triangle.

Law of the Mean for Integrals

4173 [1945, 463]. *Proposed by Herbert Robbins, United States Naval Academy, Annapolis*

Let $a<b$ be given numbers and let $f(t)$ be defined, continuous, non-negative, and strictly increasing for $a \leqq t \leqq b$. By the law of the mean for integrals, for every $p>0$ there will exist a unique number x_p, $a \leqq x_p \leqq b$, such that

$$f^p(x_p) = \frac{1}{b-a}\int_a^b f^p(t)dt.$$

Find $\lim_{p\to\infty} x_p$.

Solution by M. J. Norris, United States Naval Academy, Annapolis. Let ϵ be a given number such that $0<\epsilon<(b-a)/2$. Since $f(t)$ is strictly increasing, $f(b-\epsilon)/f(b-2\epsilon)>1$. Hence there exists a positive integer P such that for $p>P$,

$$\left[\frac{f(b-\epsilon)}{f(b-2\epsilon)}\right]^p > \frac{b-a}{\epsilon}$$

or

$$f^p(b-\epsilon) > \frac{b-a}{\epsilon} f^p(b-2\epsilon).$$

We have also

$$\int_a^b f^p(t)dt > \int_{b-\epsilon}^b f^p(b-\epsilon)dt,$$

and therefore

$$\frac{1}{b-a}\int_a^b f^p(t)dt > \frac{1}{b-a}\int_{b-\epsilon}^b f^p(b-\epsilon)dt = \frac{1}{b-a}\cdot\epsilon\cdot f^p(b-\epsilon) > f^p(b-2\epsilon).$$

Hence x_p cannot satisfy

$$f^p(x_p) = \frac{1}{b-a}\int_a^b f^p(t)dt$$

unless $x_p>b-2\epsilon$. It follows that there exists a P such that, whenever $p>P$, $x_p>b-2\epsilon$. Hence $\lim_{p\to\infty} x_p=b$.

Solved also by R. C. Buck, and the Proposer.

Buck's solution makes use of a standard theorem, Hardy, Littlewood, Pólya, *Inequalities*, p. 143. $f(x_p)=\mathfrak{M}_p(f;\ a,\ b)$ tends, as $p\to\infty$, toward the essential maximum of $f(t)$. Thus, since $f(t)$ is strictly monotonic, and continuous, and has a continuous inverse, $\lim f(x_p)=f(b)$, and $b=\lim x_p$.

Boolean Rings

4174 [1945, 463]. *Proposed by Irving Kaplansky, Harvard University*

Stone has called a ring "Boolean" if all its elements satisfy the equation $x^2=x$. Show that a ring in which $x^2=\pm x$ is either Boolean or the direct sum of a Boolean ring and the Galois field of three elements.

Solution by M. F. Smiley, United States Naval Academy. Let B be a ring satisfying $x^2=\pm x$. If $x^2=x$, then $x^3=x^2=x$; while if $x^2=-x$, then $x^3=-x^2=x$. Hence always $x^3=x$ for $x\in B$. From $(2x)^3=2x$, we obtain $6x=0$ for every $x\in B$. Let B_2 and B_3 be the ideals for which $2x=0$ for $x\in B_2$ and $3x=0$ for $x\in B_3$. Clearly B_2 is a Boolean ring and B is the direct sum of B_2 and B_3. If $B_3\neq(0)$, let $z\in B_3$, $z\neq 0$. There is, then, an element $x\in B_3$ for which $x^2=x$ and $x\neq 0$, for we may take $x=z$ if $z^2=z$ and $x=-z$ otherwise. Now if $y\neq 0$ and $y\in B_3$, then either (a) $y^2=y$, or (b) $y^2=-y$. Case (a). By direct computation $(x+y)^2+(x-y)^2$ has the value $-x-y$, while our basic assumption permits $\pm(x+y)\pm(x-y)=\pm 2x$ or $\pm 2y=\mp x$ or $\mp y$. Since both x and y are non-zero, we conclude that $y=x$. Case (b). Here $(x+y)^2+(x-y)^2$ has the value $y-x$. Consequently, since both x and y are non-zero, we have $y=-x$. It follows that $B_3=[0, x, -x]$, as was to be proved.

Solved also by M. J. Norris, Olga Taussky and John Todd, and the Proposer.

Taussky and Todd remark that this ring is commutative, and that its components B_2 and B_3 are orthogonal. The Proposer proves the former in his abstract, *Bulletin of the American Mathematical Society*, January 1945.

Intersections of a Quadric and a Tetrahedron

4178 [1945, 522]. *Proposed by N. A. Court, University of Oklahoma*

If among the twelve points of intersection of a quadric surface with the edges of a tetrahedron there are six points located on the six edges, such that the three lines joining the points on opposite edges are concurrent, the remaining six points of intersection also have this property.

Note. This is the converse of a known proposition (see, for example Baker, *Principles of Geometry*, vol. 3, p. 54, ex. 17, 1923): The transversal is drawn from a point O to each pair of opposite joins of four points A, B, C, D, so giving rise to six points. Six other points, one on each join, are similarly obtained from another point O'. The twelve points lie on a quadric.

Solution by Howard Eves, Oregon State College. We shall employ the lemma: *Given a tetrahedron $A_1A_2A_3A_4$ and six points A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}, point A_{ij} lying on the edge A_iA_j, then a necessary and sufficient condition for $A_{12}A_{34}$, $A_{13}A_{24}$, $A_{14}A_{23}$ to be concurrent in a point P is that the sets A_2A_{34}, A_3A_{24}, A_4A_{23}; A_1A_{34}, A_3A_{14}, A_4A_{13}; A_1A_{24}, A_2A_{14}, A_4A_{12}; A_1A_{23}, A_2A_{13}, A_3A_{12} be respectively concurrent in points P_1, P_2, P_3, P_4.*

The necessary part of the lemma is readily established by considering projections of the entire figure from the vertices onto the opposite faces. This same projection procedure shows that the sufficiency condition insists that A_1P_1, A_2P_2, A_3P_3, A_4P_4 are either concurrent or else constitute a hyperbolic set of lines. The second possibility, however, is eliminated by the fact that A_1P_1 A_2P_2 must intersect, since both lie in the plane $A_1A_{34}A_2$. This guarantees that $A_{12}A_{34}$, $A_{13}A_{24}$, $A_{14}A_{23}$ are also concurrent.

We now establish the given theorem for the case where the quadric is a

sphere, noting that the general case follows from this one by projection. Let the sphere cut the edge A_iA_j in the two points A_{ij}, B_{ij}, and assume that $A_{12}A_{34}$, $A_{13}A_{24}$, $A_{14}A_{23}$ are concurrent in a point P. Then, by the lemma, the sets A_2A_{34}, A_3A_{24}, A_4A_{23}; $\cdots$ are concurrent in points P_1, P_2, P_3, P_4. It therefore follows that (see, for example, art. 227 in Johnson's *Modern Geometry*) B_2B_{34}, B_3B_{24}, B_4B_{23}; $\cdots$ are concurrent in points Q_1, Q_2, Q_3, Q_4. Hence, by the lemma again, $B_{12}B_{34}$, $B_{13}B_{24}$, $B_{14}B_{23}$ are concurrent in a point Q.

Solved also by the Proposer who refers to a theorem in his *Modern Pure Solid Geometry*, p. 119, art. 348, and also to a theorem in the *Nouvelles Annales de Mathématiques*, 1842, p. 403, and due to O. Terquem, cofounder and editor.

Coplanar Radical Axes

3946 [1940, 115]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a tetrahedron $ABCD$, and an arbitrarily chosen point P: (1) Show that the radical planes of an arbitrary sphere through P with each of the spheres $(PBCD)$, $(PCDA)$, $(PDAB)$, $(PABC)$, circumscribing the tetrahedrons in the parentheses, meet respectively the planes of the faces BCD, CDA, DAB, ABC, in four lines lying in a plane. (2) If an arbitrary plane cuts the planes of the faces BCD, CDA, DAB, ABC in the straight lines Δ_a, Δ_b, Δ_c, Δ_d, show that the planes (P, Δ_a), (P, Δ_b), (P, Δ_c), (P, Δ_d) cut the spheres $(PBCD)$, $(PCDA)$, $(PDAB)$, $(PABC)$, respectively, in four circles of a single sphere through P.

Solution by R. Bauvaist, Vincelles, Saône-et-Loire, France

(1) Let S be the sphere $(ABCD)$ and Σ the arbitrary sphere through P. If the radical plane of S and Σ intersects the plane BCD in the line Δ_a, then every point of Δ_a has the same power with respect to S, Σ, and the sphere $(PBCD)$. It follows that the four lines Δ_a, Δ_b, Δ_c, Δ_d are the four lines in question, and they all lie in the radical plane of S and Σ.

(2) Conversely, π being the plane containing Δ_a, Δ_b, Δ_c, Δ_d, there exists a sphere Σ passing through P and having π as radical plane with sphere S. This sphere Σ passes through the points of intersection of Δ_a, Δ_b, Δ_c, Δ_d with the spheres $(PBCD)$, $(PCDA)$, $(PDAB)$, $(PABC)$, whence the four circles of intersection of the four spheres with the planes (P, Δ_a), (P, Δ_b), (P, Δ_c), (P, Δ_d) all lie on Σ.

Row Sums of a Matrix

4168 [1945, 400]. *Proposed by Henry Scheffé, University of California at Los Angeles*

The matrix $X=(x_{ij})$ is real $m\times n$ with $m\leqq n$. Let the ith row sum be $y_i = \sum_j x_{ij}$. The elements x_{ij} may vary subject to the conditions (1) $XX'=I$, (2) $y_1=y_2=\cdots=y_m$. Write $y_i=y$. Show that the maximum and minimum values of y are $\sqrt{n/m}$ and $-\sqrt{n/m}$ respectively.

Solution by R. C. Buck, Harvard University. Let H_k be the $1\times k$ matrix $(1, 1, \cdots, 1)$. If A is an $n\times n$ matrix, then $\sum(A)$, the sum of all the entries

of A, is given by H_nAH_n'. The condition that the sum of each of the rows of X is y may be expressed by $H_nX'=yH_m$. Let $A=X'X$. Then

$$\sum(A) = H_n(X'X)H_n' = (yH_m)(yH_m') = my^2.$$

Since $XX'=I$,

$$AA' = X'X(X'X)' = X'XX'X = X'IX = A.$$

Thus,

$$my^2 = \sum(A) = \sum(AA') = H_nAA'H_n'$$

so that

$$my^2 = (H_nA)H_n' = (H_nA)(H_nA)'.$$

Let $H_nA=(x_1, \cdots, x_n)$ be a point of n-space. Our equation then becomes

$$my^2 = \sum_1^n x_i = \sum_1^n x_i^2,$$

and $\sum_1^n(x_i^2-n_i)=0$ is the equation of a sphere, center $(\frac{1}{2}, \cdots, \frac{1}{2})$, passing through the origin. $\sum_1^n x_i^2$ is the square of the distance from P to the origin, and is greatest when P is diametrically opposite to the origin, this value being n. Thus $my^2 \leqq n$ and $|y| \leqq n/m$.

The restriction $m \leqq n$ seems unnecessary.

Regular Polygons

4086 [1943, 391: 1944, 480]. Corrected. *Proposed by Paul Erdös, University of Michigan*

Let $A_1, A_2, \cdots, A_{2n+1}$ be the vertices of a regular polygon and let O be any point in its interior. Show that at least one of the angles A_iOA_j satisfies the relation:

$$\pi\left(1-\frac{1}{2n+1}\right) \leqq A_iOA_j \leqq \pi.$$

Solution by C. R. Phelps, Rutgers University. The stated proposition is true for any number of sides, even or odd, and hence will be shown for a regular n-gon.

Let A_1 be a vertex such that $OA_1 \leqq OA_j$ for all $j \neq 1$. The lines A_1A_j divide the interior of the polygon so that O lies in, or on the boundary of, some triangle $A_1A_kA_{k+1}$. If O lies on such a boundary, the theorem follows trivially: $A_1OA_k=\pi$. Otherwise, O lies inside the triangle $A_1A_kA_{k+1}$, and the angles $A_1OA_k<\pi$ and $A_1OA_{k+1}<\pi$. Also, since $OA_1 \leqq OA_j$, angles $OA_kA_1 \leqq OA_1A_k$ and $OA_{k+1}A_1 \leqq OA_1A_{k+1}$. But $OA_1A_k+OA_1A_{k+1}=\pi/n$. Thus $A_1OA_k+A_1OA_{k+1}=2\pi-(OA_1A_k+OA_1A_{k+1})-(OA_kA_1+OA_{k+1}A_1) \geqq 2\pi-\pi/n-\pi/n=2\pi(1-1/n)$.

Therefore, either $A_1OA_k \geqq \pi(1-1/n)$ and the theorem is proved, or else $A_1OA_k<\pi(1-1/n)$ and, by subtraction, $A_1OA_{k+1} \geqq \pi(1-1/n)$.

Solved also by Robert Steinberg and the Proposer.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

College Algebra. By M. W. Keller. Boston, Houghton Mifflin Co., 1946. 12+471 pages. $3.00.

In examining a text like Professor Keller's, one looks, first, for clarity and rigor in the subject matter itself and, second, for a certain liveliness and pedagogical insight in the manner of presentation. (Or perhaps I should list the qualities in reverse order!) Either attribute without the other, in an elementary text, is largely lost. And, considering the tremendous possibilities of the two, I think that new elementary texts still can be greeted with eagerness and serious judgement.

Mr. Keller's *College Algebra* has many unique and fine characteristics to recommend it. The contents include, in addition to the standard, basic topics of college algebra, chapters on Theory of Equations (including Horner's Method and solutions for the general cubic and general quartic), Determinants (including expansion by minors and solutions to n homogeneous linear equations in n unknowns), Mathematics of Finance, Probability and Insurance, Partial Fractions, and Infinite Series. There is no unusual overlapping of topics or anticipation of more advanced mathematics courses.

The clarity of the discussion is at once impressive. There is no hesitation in giving even more detailed explanations than are customary. Indeed, even *incorrect* manipulations are anticipated and pointed out. (Whether psychologists would disagree with the display of wrong cancellations, I do not know—but no one who has had much experience teaching college algebra would disagree!) The presentation of the long division process for polynomials is certainly self-sufficient; the distinction between dependent and inconsistent linear equations is made simple and straightforward; and, particularly, I like the development of mathematical induction through specific examples. The diagrams are streamlined and simple, yet not lacking in essential detail.

Rigor, as well, is given conscientious attention, with the result that several customary pitfalls are neatly avoided. For instance, in rules for operations with exponents, the generalization from positive rational integral to fractional exponents is careful and explicit. The necessity of the check in the solution of fractional equations is emphasized. And, all the way through, pointed distinctions are drawn between general proofs and specific examples.

But it is in the pedagogical insight of the book that Mr. Keller hits his highest stride. It is certainly a pleasure to discover, already written out, things which instructors inevitably find themselves writing out, over and over again. For instance, the derivation of the synthetic division process from long division is given fully, step by step. And as an introduction to the law $b^n \div b^m = b^{n-m}$, the terms in

$5^6 \div 5^2$ are written out completely. Then, when logarithms are presented (not, I regret, immediately after exponents), the logarithms appear, in the first few examples, actually written as exponents on 10. This is indeed refreshing! And there are other such welcome innovations.

It is against the most intangible quality of all—the liveliness and persuasiveness of the presentation—that I levy my only adverse criticism. Particularly in the somewhat stilted beginning of each chapter is lost a real chance for capturing the student's interest. The initial remarks that are made have precision and orderliness—qualities which would attract the graduate mathematics student; but to the undergraduate, they would seem lifeless and uninteresting. (I would not, for this reason, shrink for one moment from the use of mature mathematical vocabulary.) It is difficult for the beginning student to understand the general statement of what is to come unless he first sees specific examples of it. By introducing earlier in the various chapters such interesting examples as occur later, the author could both stimulate initial interest in each topic and give even more pointed indication of the immense and sweeping applicability of the science. The reviewer may appear finical, but this is no trivial matter; it is the "small spark of inspiration, which makes all the difference in the world."

However, let not this one objection detract from the value of the book. All in all, it is a unique and excellent newcomer to the catalogue of college algebra texts.

A. F. STREHLER

College Mathematics; A General Introduction. By C. H. Sisam. New York, Henry Holt and Co., 1946. 13+561 pages. $3.50.

As stated in the preface "this text presents the customary first-year course in college algebra, trigonometry, and analytic geometry, together with the notation and elementary processes and applications of the differential and integral calculus."

Except for the calculus there is ample material in the book for a separate course in each of these subjects, although in places the treatment is necessarily brief; the needs of these separate courses, or of a combined course, are admirably met. The derivative is introduced in the chapter on tangents and normals and the discussion is carried through the study of maxima and minima. Integration is covered in the same chapter, through its application to finding area. The average student, it would appear, might find difficulty with this section.

The book does not offer a "unified" course. For the most part the subjects are presented in distinct units, but when it is thought desirable, a topic may be presented in various parts. For example, polar coordinates are introduced after the treatment of the circle, and continued at this stage through the general polar equation of a circle; the polar equation of a conic comes later, in the chapter in conic sections; and then in the chapter on the graph of an equation, polar coordinates are resumed in this connection. This division seems to make the topic fit more naturally into the scheme of things. Further, in this vein, this same

chapter on the graph of an equation comes after that on the calculus, and use is made of slopes, maxima and minima, *etc.*

A highly noteworthy feature of the book is the accuracy of the definitions and explanations. To mention a few of the former, there are the definitions of rational number, given early, of the equation of a locus, and of an extraneous root of an equation, all stated explicitly. For the latter may be mentioned the section on computations involving zero, the statement of the terminology of proportions, and the discussion of inverse trigonometric functions. These, and many others, should reduce the number of students' questions to a minimum.

The exercises are sufficient in number, of varying complexity for selection by the instructor, with answers furnished for the odd-numbered ones, and those for the even-numbered to be printed separately.

There are approximately one hundred pages of tables, with explanations for their use.

A. SPITZBART

Scientific, Medical, and Technical Books Published in the United States of America 1930–1944; A Selected List of Titles in Print with Annotations. Edited by R. R. Hawkins, Washington, National Research Council, 1946. 15+1114 pages. $20.00.

"This volume contains a selected list of medical, scientific and technical books by citizens of this country and Canada published within the limits of the continental United States since 1930—books still in print and available for both domestic and foreign distribution." Approximately six thousand works are described, including 424 on mathematics, 211 on physics and 60 on astronomy. The complete title, table of contents, publisher and price of each book is supplemented by a brief descriptive note. For the most part these annotations are written with care and restraint to emphasize significant or unusual features of a book and also to indicate any special use or purpose. The roster of recent mathematical books is quite complete, although a few well known titles are not listed—perhaps because they were temporarily out of print when the volume was compiled. Noteworthy features are an author index, a subject index and a handy directory of publishers.

B. H. COLVIN

NEW BOOKS RECEIVED

Introduction to College Mathematics. By C. V. Newsom. New York, Prentice-Hall, Inc., 1946. 8+344 pages. $4.65 trade price, $3.50 text price.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1945-46

Pi Mu Epsilon, University of Georgia

The Georgia Alpha Chapter of *Pi Mu Epsilon* has been most fortunate in having Dr. Tomlinson Fort, Director General, *Pi Mu Epsilon* Fraternity, as Head of the Mathematics Department of the University of Georgia. Dr. Fort and Dr. D. F. Barrow, Faculty Director of the local chapter, have worked together in securing several very prominent mathematicians for *Pi Mu Epsilon* programs.

Fermat's last theorem, by Dr. Archibald Henderson of the University of North Carolina—a chapel program.

Fundamental theorem of arithmetic, by Dr. H. S. Vandiver of the University of Texas.

Applications of geometry, by Dr. J. M. Thomas of Duke University.

Applied mathematics, by Dr. R. G. D. Richardson of Brown University.

Topics discussed by members of the local chapter were:

Euler's transformation, by Dr. T. Fort and Dr. D. F. Barrow.

Theory of numbers, by Dr. R. J. Levit.

Nomographs, by Mr. William Burke.

Graphical solution of spherical triangles, by Dr. D. F. Barrow.

Mathematics for naval air cadets, by Mr. P. A. De Vore.

Development of scientific organizations with particular emphasis on mathematical organizations, by Dr. T. Fort.

Two initiation and social meetings were held in December and March at Dr. Fort's home.

Sponsored by *Pi Mu Epsilon*, the Pythagorean Freshman Club was organized in January. The purpose is to promote an appreciation of mathematics.

Officers for 1945–1946 were: President, Catherine Littlejohn; Vice-President, Jean Hannmack; Secretary-Treasurer, Bob Jones; Corresponding Secretary, Iris Callaway; Faculty Director, Dr. D. F. Barrow. Officers for 1946–1947 are: President, Marquilla Stuckey; Secretary-Treasurer, Frances Hammond; Corresponding Secretary, Iris Callaway; Faculty Director, Dr. R. J. Levit.

Kappa Mu Epsilon, Albion College

The *Michigan Alpha* Chapter of *Kappa Mu Epsilon* had many interesting meetings during the past year. One in particular centered about the history of Albion's *Mathematics Club*. For roll call each member gave or read a letter containing an autobiography of some woman member of this organization who had, since graduation, done graduate work.

Other topics for roll call were tricks or puzzles in mathematics, contributions

made to trigonometry, and what each member personally wished to do with his background in mathematics.

Two movies constituted one program, and papers were presented at other meetings on the following subjects:

The use of logarithms in addition and subtraction
Colorimetry
The V-T fuse
Women in mathematics
Casting counters and counting boards
Philosophy of mathematics
Scales of notation
Nomographs.

Two veterans, Robert Maynard and Phil Sawyer, returned to join the group composed of Janis Barker, Marion Bunte, Ruth Helzer, Harriette Leonard, Audrey McPherson, Charles Parkhurst, Audrey Schuett, Mary Shattuck, Amy Thomas, Dr. E. E. Ingalls, and Dr. E. R. Sleight.

The new members initiated during the year were: Jean Moffett, Irwin Weber, Nell Barton, Olive Conway, Faye Engstrom, William Hopkins, Dorothy Manley, John Nizon, Lucille Richardson, Shirley Searles, Eleanor Harper, and Mrs. Myrtle Shattuck.

Officers during the year were: President, Mary Shattuck; Vice-President, Harriette Leonard; Secretary-Treasurer, Marion Bunte; Program committee member, Janis Barker.

Mathematics Club, University of Virginia

There were five meetings of the elementary *Mathematics Club* with the following speakers and subjects:

Mathematical puzzles, by W. R. Utz
Some problems in number theory, by E. E. Floyd
History and properties of the number π, by E. V. N. Goetchius
Some problems in maxima and minima, by P. B. Buck.

Kappa Mu Epsilon, Chicago Teachers College

The programs and events recently completed began with an orientation meeting at which refreshments were served. At subsequent meetings the following topics were discussed:

Use of the slide rule
Nomography, by Dr. J. S. Georges, of Wright Junior College
Non-euclidean geometry, by Dr. J. Sachs
Spherical trigonometry, by Miss Mary Kirkpatrick
Must a mathematician teach?, by Professor J. H. Zant of Oklahoma A. & M. College. Eighty-three persons were present at this interesting meeting.

Thirty-three people attended a dinner party at which kodachrome slides and movies of the "Rockies" and the Northwest were shown. The high light of another meeting was the presentation of a token to Dr. J. T. Johnson upon his

retirement from the mathematics department. At the initiation party twenty-three new members were initiated into *Illinois Gamma* Chapter of *Kappa Mu Epsilon*.

At present each member is preparing to present a problem in which he is especially interested. These problems are those not usually found in class room work, are not too difficult, and may be understood by the membership. Art students have been interested in making posters advertising the meetings and frequently have succeeded in portraying novel ideas concerning certain aspects of mathematics. Articles are being prepared by some of the student members for publication in *The Pentagon*.

The officers for the year 1946–47 are: President, Cloda Augelli; Vice-President, Joseph Duffy; Secretary, Mary Therese Graham; Treasurer, Dolores Grien; Faculty Sponsor, Professor J. J. Urbancek.

Mathematics Club, Oberlin College

The Oberlin College Mathematics Club held nine regular meetings in 1945–46. Refreshments were served at each meeting after which a paper was presented by a student. Each paper was submitted to a faculty committee to be judged for the Mary Emily Sinclair prize. The following papers were presented:

Scales of notation—Artha Jean Burington
Schwarz paradox—Jerry Howald
Space filling curves—Mary Wright
Continued fractions—Charlotte Kessler
The Apollonius circle problem—Rodney Hood
Pascal's theorem and airplane design—Margaret Waugh
Solving maximum and minimum problems without calculus—Mary Kinsman
Probability and pi—Rosalind Monastersky and Ruth Berger
Rollers—Frank Marzocco.

In addition to the regular meetings two special meetings were held. Professor and Mrs. J. F. Randolph invited the club to their home for a Christmas party. After a social hour, Jerry Howald and Rodney Hood demonstrated space curves by intersecting string models with planes or cones of light from a projector. In the spring a banquet was held; upon that occasion the members solved mathematical puzzles, and Dr. R. S. Burington of Washington, D. C., discussed the role of equivalence in pure and applied science. Dr. Burington used illustrative material from his experience as an applied mathematician in industry and in governmental work.

The officers for the year were: President, Rodney Hood; Vice-President, Mary Kinsman; Secretary-Treasurer, Charlotte Peters; Social Chairman, Margaret Waugh; Publicity Chairman, Jerry Howald; Faculty Adviser, Professor J. F. Randolph.

NEWS AND NOTICES

EDITED BY B. W. JONES, Cornell University

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

THE NATIONAL MATHEMATICS MAGAZINE

The *National Mathematics* magazine has been revived, and will be published as a bi-monthly, except in June-July. It will feature expository articles designed to acquaint its readers with modern researches in specialized fields.

Immediate publication of research papers in pure and applied mathematics is planned. To make this possible from the standpoint of time, refereeing will be replaced by critiques from editors and others published in a department devoted to this purpose, with answers by authors, as is done in some of the engineering magazines. Professor H. V. Craig of Texas will have charge of this department. To make it financially possible authors will have their choice of awaiting their turn or paying for extra pages subject to possible return of their money.

The magazine will be under the management of D. H. Hyers of U.S.C., Glenn James of U.C.L.A., and A. D. Michal of California Institute of Technology. It will be financed by subscriptions of $3.00 per year and sponsor-contributions. These should be sent to Glenn James, University of California at Los Angeles, 405 Hilgard Avenue, Los Angeles, California.

DUTCH MATHEMATICIANS

The following deaths are reported: Drs. M. J. Belinfante and M. M. Biedermann of Amsterdam, Professor Otto Blumenthal of the Technical School of Aachen (since 1939 a refugee in Delft), Professor Julius Wolff and Dr. L. W. Nieland of the University of Utrecht, Dr. R. Remak (formerly Privatdozent at the University of Berlin and since 1939 a refugee in Amsterdam), Professor G. Schaake of the University of Groningen. All but the last died in consequence of German activities during the occupation.

The following have retired: Professors J. A. Schouten and J. F. Schuh of the Technical School of Delft, Professor W. van der Woude of the University of Leiden, Professor J. Barrau of the University of Utrecht.

At Amsterdam University: Professor J. G. van der Corput of the University of Groningen and Professor D. van Dantzig of the Technical School of Delft have been appointed to professorships, Dr. E. W. Beth has been appointed to an associate professorship of logic and the philosophy of science; Drs. G. H. A. Grosheide and J. Haantjes have been appointed to lectureships at the Free University of Amsterdam. Professor G. Mannoury has been awarded the honorary degree of Ph.D.

At the Technical School of Delft: Drs. N. G. de Bruyn, S. C. van Veen and C. Visser have been appointed to professorships.

At Groningen University, Drs. J. H. C. Gerretsen and C. S. Meyer have been promoted to professorships.

At the University of Utrecht Dr. H. Freudenthal has been appointed to a professorship.

The Mathematical Centre, an institution for the promotion of pure and applied mathematics, was founded in Amsterdam on February 11, 1946.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The twenty-fifth annual meeting of the N.C.T.M. will be held at Chalfonte-Haddon Hall, Atlantic City, New Jersey, on February 28 and March 1, 1947. The program may be found in the January number of *The Mathematics Teacher*.

HARRY BATEMAN RESEARCH FELLOWSHIP

The California Institute of Technology announces the creation of a research fellowship in pure mathematics to be known as the Harry Bateman Research Fellowship. Candidates should have obtained the Doctorate, or expect to receive it prior to the beginning of the academic year, 1947–1948. The award will be based on promise of independent research in any field of pure mathematics. The recipient will devote the major part of his time to research. In addition he will be expected to teach one advanced course in mathematics. The stipend will be $3000 for the academic year. The appointment is normally for one year, but may be renewed for a second year. Application blanks may be obtained from the Dean of the Faculty, California Institute of Technology, Pasadena 4, California, and must be returned to the same address before March 15, 1947. The appointment will be announced by April 1, 1947.

PERSONAL ITEMS

Professor L. V. Ahlfors of the University of Zurich has been appointed to a professorship at Harvard University.

Professor A. A. Albert of the University of Chicago will be a visiting professor at the University of Brazil in 1947.

Assistant Professor R. H. Bardell of the University of Wisconsin in Milwaukee has been promoted to an associate professorship.

Dr. Stefan Bergman has been appointed a lecturer at Harvard University.

Professor Emeritus Felix Bernstein of New York University has been appointed lecturer at Triple-Cities College, Syracuse University, Endicott, New York.

Assistant Professor M. A. Biot of Columbia University has been appointed to a professorship at Brown University.

Associate Professor A. W. Boldyreff of the University of Arizona has been appointed to an associate professorship at Wittenberg College.

Professor J. A. Caparó of Notre Dame University has retired with the title of professor emeritus in electrical engineering.

Dr. J. B. Coleman has been appointed acting associate professor at the University of Georgia.

Dr. R. W. Cowan of the University of Alabama has been promoted to an assistant professorship.

Assistant Professor H. R. C. Dieckmann of Occidental College has been appointed to an assistant professorship at San Jose State College.

Professor H. H. Ferns of the University of Saskatchewan has been appointed head of the department of mathematics.

Dr. A. D. Fialkow has been appointed an adjunct professor at Brooklyn Polytechnic Institute.

Walter Fleming has been appointed lecturer at the University of Manitoba.

Adjunct Professor Mary L. Foster of the University of South Carolina has been appointed to an associate professorship at Henderson State College, Arkadelphia, Arkansas.

N. D. Griffin of Oklahoma Agricultural and Mechanical College has been promoted to an assistant professorship.

Dr. C. C. Grove has been appointed a special lecturer at the University of Pennsylvania.

F. C. Hall has been appointed to an assistant professorship at Manhattan College.

Assistant Professor F. E. Hohn of Guilford College, Guilford, North Carolina, has been appointed to an assistant professorship at the University of Maine.

Associate Professor Grace M. Hopper of Vassar College has resigned to accept a position in the Cruft Research Laboratory at Harvard University.

Dr. A. S. Householder has accepted a position as principal physicist at the Clinton Laboratories, Monsanto Chemical Company, Knoxville, Tennessee.

Dr. Waclaw Kozakiewicz of the University of Saskatchewan has been promoted to an assistant professorship.

Professor C. C. MacDuffee of the University of Wisconsin has been appointed visiting professor of mathematics at the University of Puerto Rico.

Assistant Professor A. B. Mewborn of the University of Arizona has been appointed to an associate professorship at the postgraduate school of the United States Naval Academy.

Associate Professor R. S. Pate of the University of South Carolina has been appointed to a professorship at Michigan State Normal College, Ypsilanti, Michigan.

Dr. Echo D. Pepper has been appointed to an assistant professorship at the University of Illinois.

Associate Professor H. R. Phalen of the College of William and Mary has been promoted to a professorship.

Assistant Professor Adrienne S. Rayl of the Birmingham Center of the University of Alabama has been promoted to an associate professorship.

Professor Marcel Riesz of the Lunds Universitets Matematiska Institution, Lund, Sweden, has been appointed visiting lecturer at the University of Chicago.

Assistant Professor R. F. Rinehart of Case School of Applied Science has been promoted to a professorship.

Professor W. H. Roever of Washington University has retired with the title emeritus.

Senior Professor R. E. Root of the Postgraduate School of the United States Naval Academy has retired with the title of professor emeritus.

Dr. O. H. Schmidt has been appointed librarian of the Danish Technical Library, Copenhagen, Denmark.

L. R. Shobe, Technical Instructor at General Motors Institute, has been appointed head of the department of mathematics at State Teachers College, Bemidji, Minnesota.

W. H. Simons has been appointed to a lectureship at the University of British Columbia.

Associate Professor M. F. Smiley of Lehigh University has been appointed to an associate professorship at Northwestern University.

R. E. Smith of Allegheny College has been appointed to an associate professorship at the College of William and Mary.

Assistant Professor S. S. Smith of the University of Utah has been promoted to an associate professorship.

Associate Professor Andrew Sobczyk of Oregon State College has been appointed Chief of the Mathematical Research Branch, Watson Laboratories, Cambridge, Massachusetts.

Dr. Ruth W. Stokes has been appointed to an assistant professorship at Syracuse University.

Dr. Mildred M. Sullivan of Queens College, Flushing, New York, has been promoted to an assistant professorship.

Associate Professor P. M. Swingle of the New Mexico College of Agriculture and Mechanical Arts has been appointed to a professorship at the University of Miami.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

ANNUAL MEETING OF THE NEBRASKA SECTION

The twenty-second meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska College of Medicine in Omaha, on Saturday, May 4, 1946. Professor F. S. Harper, Chairman of the Section, presided.

The attendance was twenty, including the following thirteen members of the Association: E. M. Berry, A. K. Bettinger, W. C. Brenke, C. C. Camp, H. M. Cox, J. A. Daum, J. M. Earl, M. G. Gaba, C. B. Gass, F. S. Harper, J. F. Heyda, Sigurd Mundhjeld, and Lulu L. Runge.

At the business meeting the following officers were elected for the coming year: Chairman, Ralph Hull, University of Nebraska; Vice-Chairman, C. B. Gass, Nebraska Wesleyan University; Secretary, Lulu L. Runge, University of Nebraska.

The following papers were presented:

1. *On a certain definite integral*, by Professor M. A. Basoco, University of Nebraska.

This paper was ready by title.

2. *Some special types of generating functions of polynomial systems*, by Professor W. C. Brenke, University of Nebraska.

In this paper Professor Brenke discussed types of generating functions which arise when the function $f(xt)$ in $g(x, t)=e^t f(xt)$ is subjected to special restrictions. It was shown how this generating function gives rise to a broad class of polynomials $Y_n(x)$ among which are the generalized Laguerre polynomials, as shown in his paper in this MONTHLY, vol. 52, 1945. That none of the other classical orthogonal polynomials are among the $Y_n(x)$ was shown by proving that their generating functions can never take the form $e^t f(xt)$.

3. *Integral solutions (with the restrictions) of the equation* $x^2+y^2=z^2$, by Dr. E. M. Berry of Nebraska State Teachers College at Chadron.

The speaker dealt with solutions of the equation $x^2+y^2=z^2$ which satisfy one of the equations $|y-kx|=h$, $|z-kx|=h$, where k is any integer, or a fraction whose denominator is 2, and h is a particular integer.

4. *A fundamental equation in the mathematics of finance*, by Professor F. S. Harper, University of Nebraska.

It was pointed out that the assumptions that money is productive and that capital must not be impaired give rise to a fundamental equation of value that is used to unify the study of many of the topics in the Mathematics of Finance, and which gives a clearer understanding of the difficult subject of depreciation.

5. *A simple method for celestial navigation*, by Professor O. C. Collins, University of Nebraska, introduced by the Chairman.

It was shown by two methods which avoid all measurement (and correction) of angles, how position has been found to within one mile on land. A single measured time interval determines uniquely both celestial coordinates of the zenith. Great circles used as lines of position are plotted by selection from precomputed data, and shown to yield an unambiguous fix. One of the two methods is available for trial in flight or at sea, with the use of a bubble sextant. The other method calls for the design of a new instrument if it is to be used off the ground.

6. *Mathematics with the air corps*, by Professor Ralph Hull, University of Nebraska.

This paper was read by title.

7. *Significant differences in grades*, by Professor C. C. Camp, University of Nebraska.

The speaker found that two students with averages of 90.624 and 90.151 for Phi Beta Kappa did not differ statistically (5% level). This seems to refute current conceptions such as the principle that persons with grade averages of 88.1 or higher should be elected and those with 87.6 or less should not be elected. It would be necessary to consider the σ_m's of such students in order to determine the difference required. It might be nearer 2%, and from the study would probably be at least 1%. Quiz averages of 92.25 and 84.125 were found to be not significantly different (5% level) for eight quizzes. The minimum difference required for students with such σ_m's would be 8.28. For classes of least variability in individual quiz grades the minimum significant difference in quiz or semester grade was found to be about 5. This leads logically to a grading system with nine passing marks if 60 is the lowest.

8. *The Euler-Maclaurin formula*, by the same author.

This formula for approximate integration when carried to first-derivative terms was recommended as being more accurate than Simpson's Rule. For summing slowly converging series, but starting with about the tenth term, this formula (reversed) is extremely practical. Alternating series with terms grouped in pairs can likewise be evaluated.

9. *The measurement and analysis of the mathematical attainment of the returning G.I.*, by Professor H. M. Cox, University of Nebraska.

Preregistration guidance examinations which are required of new students at the University of Nebraska have been designed so as to measure ability to study and interpret materials in the specific fields. Certain of the examinations prepared by the United States Armed Forces Institute have been used for this purpose. In mathematics, a special examination was prepared by representatives from the Mathematics Department, the Chemistry Department, the Bureau of Instructional Research, and other departments; this examination tests differentially performance in mathematics at several levels of preparation. Although the scores on the mathematics examination would seem to be depressed for those veterans with two or more years absence from the class room,

nevertheless very significant differences exist between groups of men with varying numbers and patterns of high school courses.

LULU L. RUNGE, *Secretary*

SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association was held at the George Washington University, Washington, D. C., on Saturday, May 11, 1946, with a morning session, luncheon, and an afternoon session. Professor J. B. Scarborough, Chairman of the Section, presided at both sessions.

The attendance was fifty including the following thirty-seven members of the Association: Archie Blake, T. A. Botts, Herman Branson, Randolph Church, Abraham Cohen, J. F. Daly, J. A. Duerksen, F. D. Faulkner, E. J. Finan, B. H. Gere, B. C. Getchell, Michael Goldberg, R. A. Good, D. W. Hall, M. A. Hyman, F. E. Johnston, Sidney Kaplan, L. M. Kells, W. D. Lambert, O. E. Lancaster, A. E. Landry, M. H. Martin, Carol V. McCamman, T. W. Moore, W. H. Norris, Jr., O. J. Ramler, J. N. Rice, R. E. Root, J. B. Scarborough, E. D. Schell, Vivian E. Spencer, Hillel Spitz, J. H. Taylor, Marian M. Torrey, J. A. Ward, G. T. Whyburn, Clement Winston.

The officers elected at the business meeting of the Section were as follows: Chairman, W. K. Morrill, Johns Hopkins University; Secretary-Treasurer, E. J. Finan, Catholic University; Members of the Executive Committee, G. A. Hedlund, University of Virginia and Dr. Archie Blake, Aberdeen Proving Ground. It was agreed to hold the next meeting on Saturday, December 7, 1946, at Johns Hopkins University.

The first three of the following papers were read at the morning session. The remaining two were read at the afternoon session. Dr. Deming's paper was read at the invitation of the Section.

1. *On a poristic system of triangles*, by Professor O. J. Ramler, Catholic University.

In his paper, Professor Ramler featured the use of isotropic coördinates to show the ease and brevity with which one could develop some of the results of Weill, Servais, and Goormaghtigh in connection with loci of notable points associated with triangles inscribed in a circle and circumscribed to a conic.

2. *An integral equation of a general metabolizing system*, by Professor Herman Branson, Howard University.

For the purpose of this paper, a general metabolizing system is one in which materials are being produced, consumed, transported, and/or stored. A mathematical description of such a system leads to an integral equation of the "Faltung" type. It is shown through examples that this equation gives interesting results when applied to first order chemical reactions, and that it is especially effective in discussing certain problems arising in biological research connected

with isotopes. The speaker emphasized the rich descriptive material in biology and chemistry for a course in integral equations.

3. *Early history of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America*, by Professor R. E. Root, Post Graduate School, U. S. Naval Academy.

Professor Root gave a brief account of the founding of the Mathematical Association in 1915 and of the Maryland-Virginia-District of Columbia Section in 1917. The matter was presented by quotations from letters found in an old file. Letters quoted were mostly from E. R. Hedrick and A. Cohen.

4. *On the structure of a cluster*, by Professor R. A. Good, University of Maryland.

The ring is a special case of a more general algebraic system called the cluster. Some important subsets contained in a cluster are its derived ring and its ideals, including the annihilator ideal. Numerous properties of clusters were illustrated by an example of a cluster of order eight.

5. *On the sampling problem in the observation of the election in Greece*, by Dr. W. Edwards Deming, Bureau of the Budget.

E. J. FINAN, *Secretary*

ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association was held at Carleton College in Northfield, Minnesota, on Saturday, May 11, 1946. Two sessions were held in the forenoon, one at luncheon, and one in the afternoon. Professors Harry Nelson, C. H. Gingrich, The Reverend G. L. Winkelmann, and Professor C. S. Carlson presided at the respective sessions.

Sixty-one persons attended the meeting, including the following twenty-two members of the Association: N. R. Amundson, K. H. Bracewell, R. W. Brink, L. E. Bush, W. H. Bussey, R. H. Cameron, E. J. Camp, C. S. Carlson, Gladys Gibbons, Sister Mary Seraphim Gibbons, C. H. Gingrich, H. W. Godderz, H. E. Hartig, C. M. Jensen, W. D. Munro, H. E. Nelson, J. M. H. Olmsted, G. C. Priester, F. J. Taylor, H. L. Turrittin, K. W. Wegner, and The Reverend G. L. Winkelmann.

The following officers were elected for the coming year: Chairman, K. H. Bracewell, Hamline University; Secretary, L. E. Bush, College of St. Thomas; Executive Committee, W. H. Bussey, University of Minnesota; W. D. Munro, University of Minnesota; C. S. Carlson, St. Olaf College.

Copies of the *Memorandum to the Secretaries of the Sections of the Mathematial Association of America* from the Committee for the Coordination of Studies in Mathematical Education were passed out to those attending the first morning session. At the business session a resolution was passed instructing the Chairman of the Section to appoint a committee to cooperate with the previously mentioned committee. The newly elected Chairman appointed the following committee for this purpose: K. W. Wegner, Chairman, W. L. Hart, E. J. Camp.

By invitation of the Executive Committee, Professor Robert H. Cameron delivered an address at the second morning session. The title of his address was *A Class of Non-linear Integral Equations which Can Be Solved by Formal Procedures.* Professor Cameron investigated the quadratic convolution equation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-s-t)f(t)p(s)dtds + p_0\int_{-\infty}^{\infty} f(x-t)f(t)dt + 2\int_{-\infty}^{\infty} f(x-t)q(t)dt + 2q_0 f(x) + r(x) = 0$$

by means of a Fourier transforms. A short résumé of the necessary Fourier transform theory was given as an introduction, and a number of Fourier transforms were calculated. The special case of the above equation

$$\int_{-\infty}^{\infty} f(x-t)f(t)dt + 2f(x) = g(x)$$

was introduced and solved completely for a few particular choices of the function $g(x)$. A case in which this equation has no absolutely integrable solution was observed. Finally, the general equation was attacked by the methods used in solving the simpler one, and a formal solution was obtained. Conditions were given under which it is known that an absolutely integrable solution must exist and be unique.

In addition to the address by Professor Cameron, the following eight papers were presented:

1. *Rational values of trigonometric functions*, by Professor J. M. H. Olmsted, University of Minnesota.

This was a discussion of a paper by Professor Olmsted published in this MONTHLY, vol. 52, 1945, pp. 507–508.

2. *Upper and lower bounds for certain exponential sums*, by Professor H. L. Turrittin, University of Minnesota.

This paper dealt with the problem of computing the greatest lower bound and the least upper bound of the function

$$F(t) = \sum_{n=1}^{n} c_n \cos(P_i(t)), \qquad -\infty < t < \infty$$

where the c's are the real constants and the P's are polynomials with real coefficients. A scheme was outlined for shifting the problem from the transcendental to the algebraic domain by using an extension of a Hardy-Littlewood theorem on Diophantine approximations.

3. *Note on the circular clamped plate with an eccentric point load*, by Professor N. R. Amundson, University of Minnesota.

The speaker considered the derivation of the Green's function for a circle for the biharmonic equation by analogy with the Green's function for Laplace's equation for the circle. Applications were made to circular thin elastic plates.

4. *Some classroom techniques*, by Professor K. W. Wegner, Carleton College.

The speaker emphasized the importance of including among the objectives of any college mathematics course the following two: (1) increasing the ability of the student to read scientific material, (2) increasing the ability of the student to make precise mathematical statements. The lecture technique does not contribute toward the achievement of these objectives. A technique was described which has been successful in attaining the desired results as well as the other objectives of the course.

5. *A set of equivalent bases for topology*, by Mr. Monroe Donsker, University of Minnesota, introduced by Professor J. M. H. Olmsted.

Taking as a starting point the axiomatic system based on the closure operation which was constructed by Kuratowski, the speaker showed that certain of the topological operations which are definable in terms of the closure operation can be used as bases for constructing axiomatic systems which would be equivalent to the system based on closure. For each such operation the axioms necessary for equivalence were exhibited and the general problem of the equivalence of two axiomatic systems were discussed.

6. *Boundary values of analytic functions*, Professor S. E. Warschawski, University of Minnesota, introduced by Professor N. R. Amundson.

The following extension of a theorem of F. Riesz in *Mathematische Zeitschrift*, vol. 18 (1923) was proved: Let $f(z)$ be a function analytic in the interior of a closed rectifiable Jordan curve C. Suppose that C_n, $n=1, 2, \cdots$, is a sequence of closed rectifiable Jordan curves in the interior of C which "converge" to C as $n \to \infty$ (that is, for every $\epsilon>0$, all the points of all C_n with a sufficiently large index n are at a distance less than ϵ from C). If for a $p>0$ and all n, the integrals $\int_{C_n}|f(z)|^p ds \leqq M$, where M is independent of n, then $f(z)$ possesses boundary values for nontangential approach almost everywhere on C and $\int_C |f(z)|^p ds$ exists. The proof was accomplished by mapping C and C_n conformally on the unit circle.

7. *The design of computing control systems*, by W. D. Munro, University of Minnesota.

A few of the general aspects of the design of computers for use in bombing, and so forth, were discussed from the point of view of replacing the variables in the problem by the variables in an equivalent system (for example, an electric network). Design of a hypothetical bombsight based on idealized conditions was used as an illustration.

8. *Geometric constructions of the third and fourth degree*, by Jacob Bearman, University of Minnesota, introduced by the Secretary.

The Nicomedes trisection of the angle and the Vieta construction for the duplication of the cube are examples of "insertion" constructions. These are special cases of the general problem of inscribing in a given conic a chord of given length, subject to the restriction that the chord produced shall pass through a given point not necessarily on the conic. The special and general problems lead to quartic equations with coefficients integral rational functions of the coefficients of the equation of the conic and of the cosine or tangent of the angle formed by the coordinate axes. The solutions of the quartics determine the "angular coefficient" of the lines satisfying the given conditions.

From the basic nature of the angle trisection and the Vieta construction, it follows that any cubic or quartic equation can be solved geometrically by ruler and compasses and at most two insertion constructions; and, conversely, any construction involving only ruler and compasses and insertion constructions leads algebraically to equations of degree not higher than the fourth.

L. E. BUSH, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-ninth Summer Meeting, New Haven, Conn., September 1–2, 1947.
Thirty-first Annual Meeting, Athens, Georgia, January 1, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Peoria, May 9–10, 1947
INDIANA
IOWA, Cedar Falls, April 18–19, 1947
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, Brooklyn, April 19, 1947
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 3, 1947
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST, Vancouver, British Columbia, April 10–11, 1947
PHILADELPHIA
ROCKY MOUNTAIN
SOUTHEASTERN, Columbia, S. C., April 18–19, 1947
SOUTHERN CALIFORNIA, Claremont, March 8, 1947
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Rochester, May 10, 1947
WISCONSIN, Madison, May, 1947

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